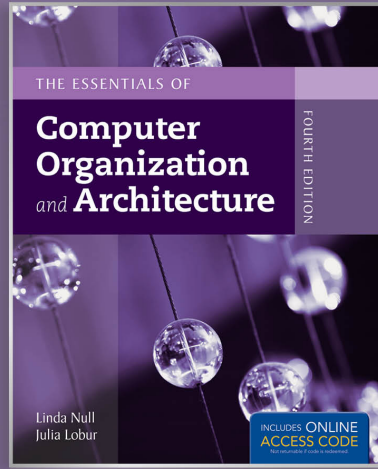


Chapter 3

Boolean Algebra and Digital Logic



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Chapter 3 Objectives

- Understand the relationship between Boolean logic and digital computer circuits.
- Learn how to design simple logic circuits.
- Understand how digital circuits work together to form complex computer systems.

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3.1 Introduction

- In the latter part of the nineteenth century, George Boole incensed philosophers and mathematicians alike when he suggested that logical thought could be represented through mathematical equations.
 - *How dare anyone suggest that human thought could be encapsulated and manipulated like an algebraic formula?*
- Computers, as we know them today, are implementations of Boole's *Laws of Thought*.
 - John Atanasoff and Claude Shannon were among the first to see this connection.

G. Boole: “*An Investigation of the Laws of Thought*” (1854)

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3.1 Introduction

- In the middle of the twentieth century, computers were commonly known as “thinking machines” and “electronic brains.”
 - Many people were fearful of them.
- Nowadays, we rarely ponder the relationship between electronic digital computers and human logic. Computers are accepted as part of our lives.
 - Many people, however, are still fearful of them.
- In this chapter, you will learn the simplicity that constitutes the essence of the machine.

John von Neumann: “*Theory of Self-Reproducing Automata*” (1966)

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3.2 Boolean Algebra

- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
 - In formal logic, these values are “true” and “false.”
 - In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low”.
- Boolean expressions are created by performing operations on Boolean variables.
 - Common Boolean operators include AND, OR, and NOT.

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3.2 Boolean Algebra

- A Boolean operator can be completely described using a **truth table**.
- The truth table for the Boolean operators AND and OR are shown at the right
- The AND operator is also known as a *Boolean product*. The OR operator is the *Boolean sum*.

X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

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3.2 Boolean Algebra

- The truth table for the Boolean NOT operator is shown at the right.
- The NOT operation is most often designated by a prime mark (X'). It is sometimes indicated by an overbar (\overline{X}) or an “elbow” ($\neg X$).

NOT X

X	X'
0	1
1	0

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3.2 Boolean Algebra

- A **Boolean function** has:
 - At least one Boolean variable,
 - At least one Boolean operator, and
 - At least one input from the set $\{0,1\}$.
- It produces an output that is also a member of the set $\{0,1\}$.

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3.2 Boolean Algebra

- The truth table for the Boolean function:
 $F(x, y, z) = xz' + y$ is shown at the right.
- To make evaluation of the Boolean function easier, the truth table contains extra (shaded) columns to hold evaluations of subparts of the function.

$$F(x, y, z) = xz' + y$$

x	y	z	z'	xz'	xz' + y
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

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3.2 Boolean Algebra

- As with common arithmetic, Boolean operations have rules of precedence.
- The NOT operator has highest priority, followed by AND and then OR.
- This is how we chose the (shaded) function subparts in our table.

$$F(x, y, z) = xz' + y$$

x	y	z	z'	xz'	xz' + y
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

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3.2 Boolean Algebra

- Digital computers contain circuits that implement Boolean functions.
- The simpler that we can make a Boolean function, the smaller the circuit that will result.
 - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

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3.2 Boolean Algebra

- Most Boolean identities have an AND (product) form as well as an OR (sum) form. We give our identities using both forms. Our first group is rather intuitive:

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$xx' = 0$	$x + x' = 1$

Idempotent: can be applied multiple times without changing the result

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3.2 Boolean Algebra

- Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity Name	AND Form	OR Form
Commutative Law	$xy = yx$	$x+y = y+x$
Associative Law	$(xy)z = x(yz)$	$(x+y)+z = x+(y+z)$
Distributive Law	$x+yz = (x+y)(x+z)$	$x(y+z) = xy+xz$

Commutative: the order can be changed without changing the result

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3.2 Boolean Algebra

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law	$x(x+y) = x$	$x + xy = x$
DeMorgan's Law	$(xy)' = x' + y'$	$(x+y)' = x'y'$
Double Complement Law	$(x)'' = x$	

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3.2 Boolean Algebra

- We can use Boolean identities to simplify:

$$F(x, y, z) = xy + x'z + yz$$

$$\begin{aligned}
 F(x, y, z) &= xy + x'z + yz && \\
 &= xy + x'z + yz(1) && \text{(Identity)} \\
 &= xy + x'z + yz(x + x') && \text{(Inverse)} \\
 &= xy + x'z + (yz)x + (yz)x' && \text{(Distributive)} \\
 &= xy + x'z + x(yz) + x'(zy) && \text{(Commutative)} \\
 &= xy + x'z + (xy)z + (x'z)y && \text{(Associative twice)} \\
 &= xy + (xy)z + x'z + (x'z)y && \text{(Commutative)} \\
 &= xy(1 + z) + x'z(1 + y) && \text{(Distributive)} \\
 &= xy(1) + x'z(1) && \text{(Null)} \\
 &= xy + x'z && \text{(Identity)}
 \end{aligned}$$

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3.2 Boolean Algebra

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- Recall DeMorgan's law states:

$$(xy)' = x' + y' \quad \text{and} \quad (x + y)' = x'y'$$

3.2 Boolean Algebra

- DeMorgan's law can be extended to any number of variables.
- Replace each variable by its complement and change all ANDs to ORs and all ORs to ANDs.
- Thus, we find that the complement of:

$$F(x, y, z) = (xy) + (x'y) + (xz')$$

is:

$$\begin{aligned} F'(x, y, z) &= ((xy) + (x'y) + (xz'))' \\ &= (xy)' (x'y)' (xz')' \\ &= (x' + y') (x + y') (x' + z) \end{aligned}$$

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3.2 Boolean Algebra

- Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.
 - These “synonymous” forms are *logically equivalent*.
 - Logically equivalent expressions have identical truth tables.
- In order to eliminate as much confusion as possible, designers express Boolean functions in *standardized* (or *canonical*) form.

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3.2 Boolean Algebra

- There are two canonical forms for Boolean expressions: sum-of-products and product-of-sums.
 - Recall the Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the **sum-of-products** form, ANDed variables are ORed together.
 - For example: $F(x, y, z) = xy + xz + yz$
- In the **product-of-sums** form, ORed variables are ANDed together.
 - For example: $F(x, y, z) = (x+y)(x+z)(y+z)$

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3.2 Boolean Algebra

- It is easy to convert a function to sum-of-products form using its truth table.
- We are interested in the values of the variables that make the function true (=1).
- Using the truth table, we list the values of the variables that result in a true function value.
- Each group of variables is then ORed together.

$$F(x, y, z) = xz' + y$$

x	y	z	xz' + y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

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3.2 Boolean Algebra

- The sum-of-products form for our function is:

$$F(x, y, z) = (x'yz') + (x'yz) + (xy'z') + (xyz')$$

We note that this function is not in simplest terms. Our aim is only to rewrite our function in canonical sum-of-products form.

$$F(x, y, z) = xz' + y$$

x	y	z	$xz' + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

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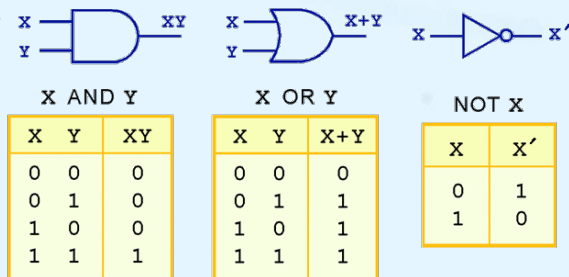
3.3 Logic Gates

- We have looked at Boolean functions in abstract terms.
- In this section, we see that Boolean functions are implemented in digital computer circuits called **gates**.
- A gate is an electronic device that produces a result based on two or more input values.
 - In reality, gates consist of one to six transistors, but digital designers think of them as a single unit. The basic physical component of a computer is the transistor; the basic logic component is the gate.
 - Integrated circuits contain collections of gates suited to a particular purpose.

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3.3 Logic Gates

- The three simplest gates are the AND, OR, and NOT gates.

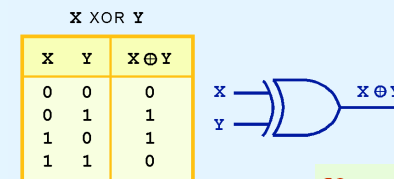


- They correspond directly to their respective Boolean operations, as you can see by their truth tables.

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3.3 Logic Gates

- Another very useful gate is the exclusive OR (XOR) gate.
- The output of the XOR operation is true only when the values of the inputs differ.

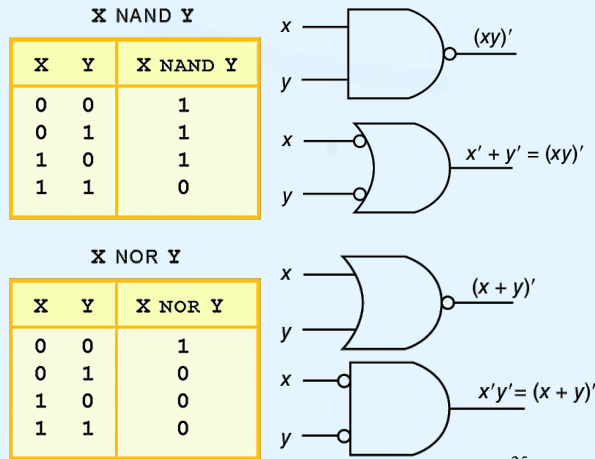


Note the special symbol \oplus for the XOR operation.

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3.3 Logic Gates

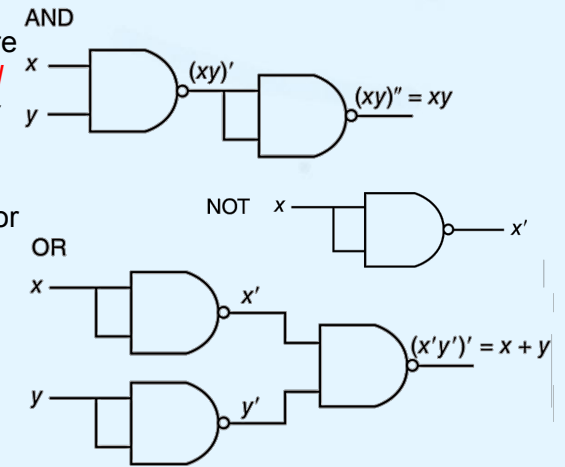
- NAND and NOR are two very important gates. Their symbols and truth tables are shown at the right.



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3.3 Logic Gates

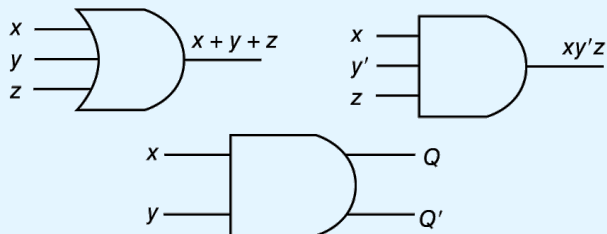
- NAND and NOR are known as *universal gates* because any Boolean function can be constructed using only NAND or only NOR gates.
- They are inexpensive to manufacture.



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3.3 Logic Gates

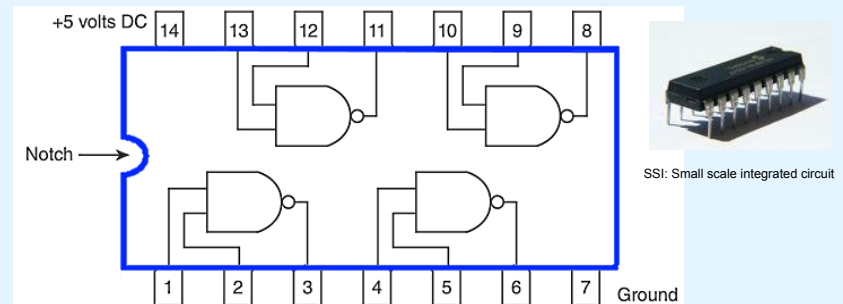
- Gates can have multiple inputs and more than one output.
 - A second output can be provided for the complement of the operation.
 - We'll see more of this later.



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3.4 Digital Components

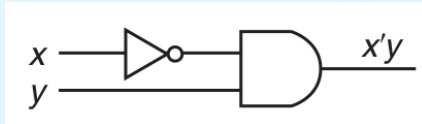
- Typically, gates are not sold individually; they are sold in units called *integrated circuits*.
- Simple SSI integrated circuit with 4 NAND gates



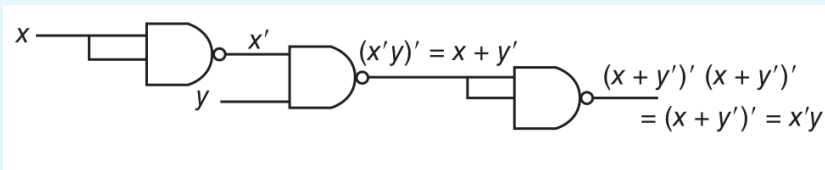
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3.4 Digital Components

- The Boolean circuit:



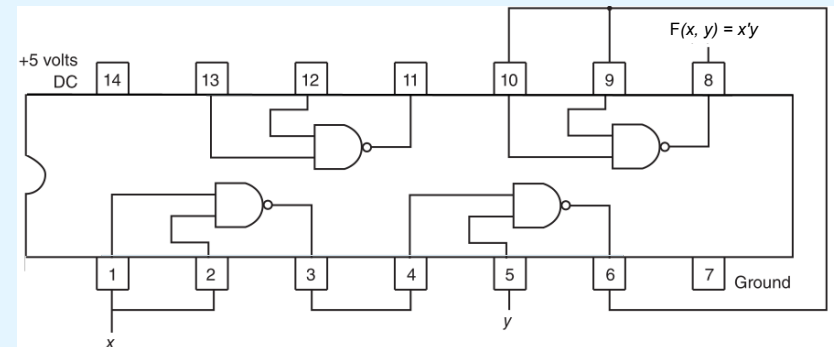
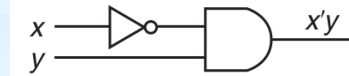
- Can be rendered using only NAND gates as:



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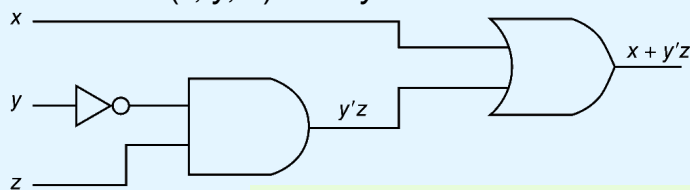
3.4 Digital Components

- So we can wire the pre-packaged circuit to implement our function:



3.4 Digital Components

- The main thing to remember is that combinations of gates implement Boolean functions.
- The circuit below implements the Boolean function: $F(x, y, z) = x + y'z$



We simplify our Boolean expressions so that we can create simpler circuits.

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3.5 Combinational Circuits

- We have designed a circuit that implements the Boolean function: $F(x, y, z) = x + y'z$
- This circuit is an example of a **combinational logic** circuit. The output is a strict *combination* of the current inputs.
- Combinational logic circuits produce a specified output (almost) at the instant when input values are applied.
 - In a later section, we will explore circuits where this is not the case.

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3.5 Combinational Circuits

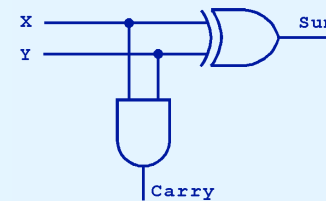
- Combinational logic circuits give us many useful devices.
- One of the simplest is the *half adder*, which finds the sum of two bits.
- We can gain some insight into the construction of a half adder by looking at its truth table, shown at the right.

Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

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3.5 Combinational Circuits

- As we see, the sum can be found using the XOR operation and the carry using the AND operation.



Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

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3.5 Combinational Circuits

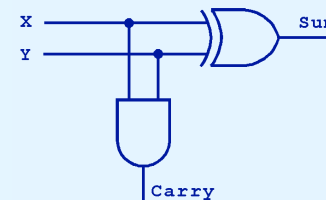
- We can change our half adder into a *full-adder* by including gates for processing the carry bit.
- The truth table for a full-adder is shown at the right.

Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

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3.5 Combinational Circuits

- How can we change the half adder shown below to make it a full-adder?

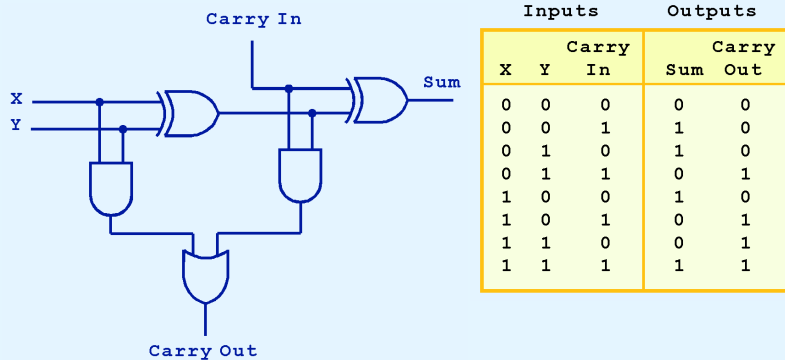


Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

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3.5 Combinational Circuits

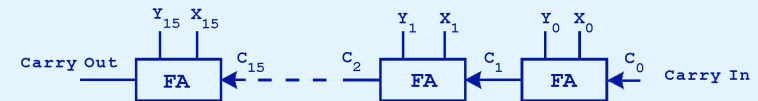
- Here is our completed full-adder (composed of two half-adders and an OR gate).



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3.5 Combinational Circuits

- Just as we combined half adders to make a full adder, full adders can be connected in series.
- The carry bit “ripples” from one adder to the next; hence, this configuration is called a *ripple-carry adder*.



Today's systems employ more efficient adders.

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3.5 Combinational Circuits

- Decoders** are another important type of combinational circuit.
- Among other things, they are useful in selecting a memory location according to a binary value placed on the address lines of a memory bus.
- Address decoders with n inputs can select any of 2^n locations.

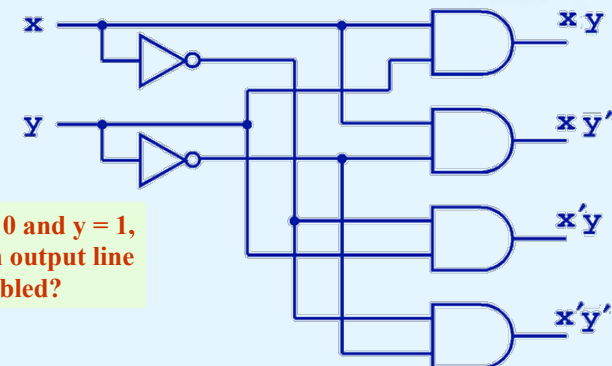
This is a block diagram for a decoder.



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3.5 Combinational Circuits

- This is what a 2-to-4 decoder looks like on the inside.

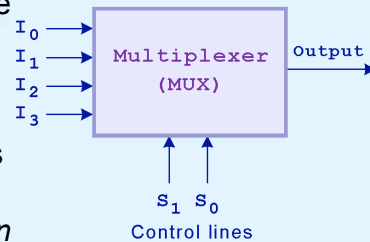


If $x = 0$ and $y = 1$, which output line is enabled?

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3.5 Combinational Circuits

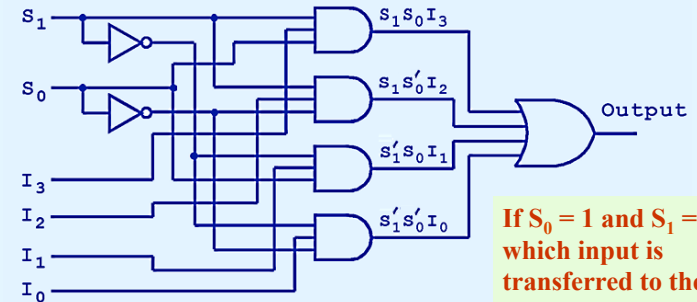
- A **multiplexer** selects a single output from several inputs.
- The particular input chosen for output is determined by the value of the multiplexer's control lines.
- To be able to select among n inputs, $\log_2 n$ control lines are needed.



This is a block diagram for a multiplexer.

3.5 Combinational Circuits

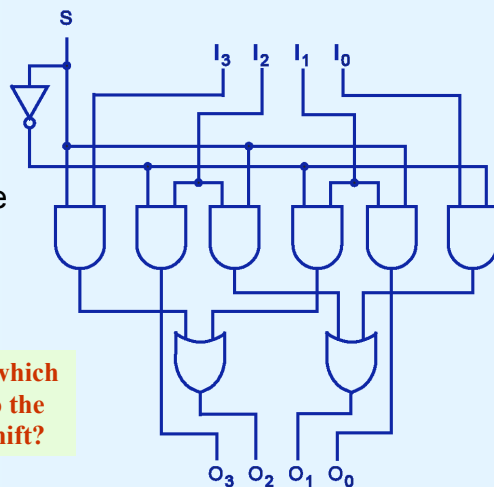
- This is what a 4-to-1 multiplexer looks like on the inside.



If $S_0 = 1$ and $S_1 = 0$, which input is transferred to the output?

3.5 Combinational Circuits

- This **shifter** moves the bits of a nibble one position to the left or right.

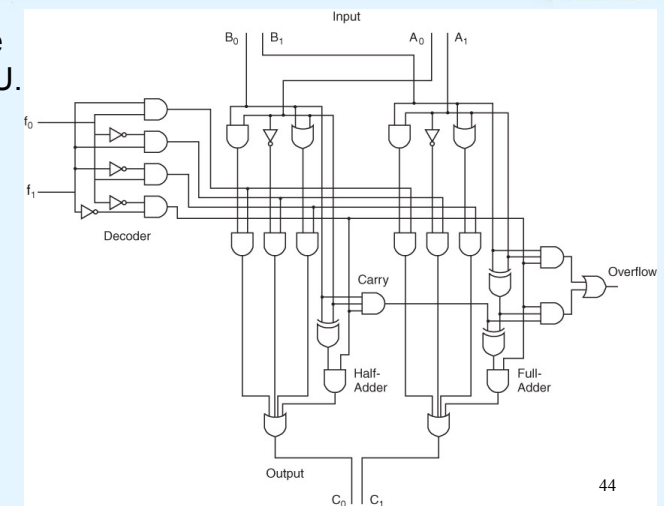


If $S = 0$, in which direction do the input bits shift?

3.5 Combinational Circuits

- A simple 2-bit ALU.

- 00: $A + B$
- 01: NOT A
- 10: $A \text{ OR } B$
- 11: $A \text{ AND } B$



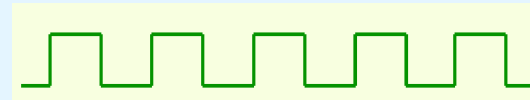
3.6 Sequential Circuits

- Combinational logic circuits are perfect for situations when we require the immediate application of a Boolean function to a set of inputs.
- There are other times, however, when we need a circuit to change its value with consideration to its current state as well as its inputs.
 - These circuits have to “remember” their current state.
- **Sequential logic circuits** provide this functionality for us. Some outputs may depend on past inputs (the *sequence* of inputs over time).

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3.6 Sequential Circuits

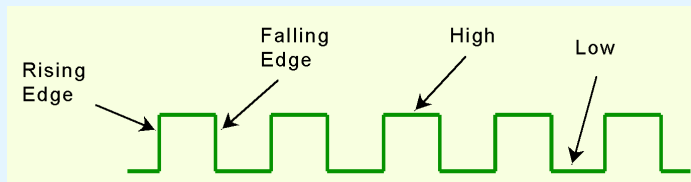
- As the name implies, sequential logic circuits require a means by which events can be sequenced.
- State changes are controlled by clocks.
 - A “clock” is a special circuit that sends electrical pulses through a circuit.
- Clocks produce electrical waveforms such as the one shown below.



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3.6 Sequential Circuits

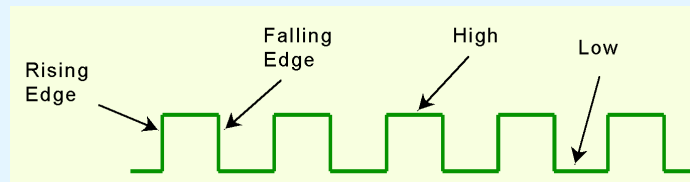
- State changes occur in sequential circuits only when the clock ticks.
- Circuits can change state on the rising edge, falling edge, or when the clock pulse reaches its highest or lowest voltage.



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3.6 Sequential Circuits

- Circuits that change state on the rising edge, or falling edge of the clock pulse are called **edge-triggered**.
- **Level-triggered circuits** change state when the clock voltage reaches its highest or lowest level.

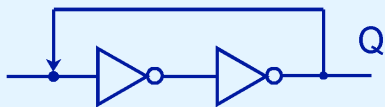


Most sequential circuits are edge-triggered.

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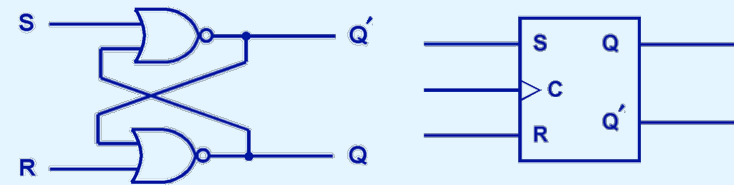
3.6 Sequential Circuits

- To retain their state values, sequential circuits rely on **feedback**.
- Feedback in digital circuits occurs when an output is looped back to the input.
- A simple example of this concept is shown below.
 - If Q is 0 it will always be 0, if it is 1, it will always be 1. Why?



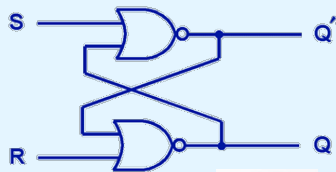
3.6 Sequential Circuits

- You can see how feedback works by examining the most basic sequential logic components, the **SR flip-flop**.
 - The “SR” stands for set/reset.
- The internals of an SR flip-flop (using 2 NOR gates) are shown below, along with its block diagram.



3.6 Sequential Circuits

- The behavior of an SR flip-flop is described by a characteristic table.
- $Q(t)$ means the value of the output at time t . $Q(t+1)$ is the value of Q after the next clock pulse.



S	R	$Q(t+1)$
0	0	$Q(t)$ (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	undefined

X NOR Y		
X	Y	X NOR Y
0	0	1
0	1	0
1	0	0
1	1	0

3.6 Sequential Circuits

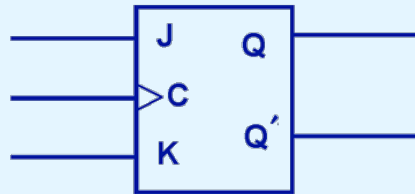
- The SR flip-flop actually has three inputs: S, R, and its current output, Q .
- Thus, we can construct a truth table for this circuit, as shown at the right.
- Notice the two undefined values. When both S and R are 1, the SR flip-flop is unstable.

Present State			Next State
S	R	$Q(t)$	$Q(t+1)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	undefined
1	1	1	undefined

3.6 Sequential Circuits

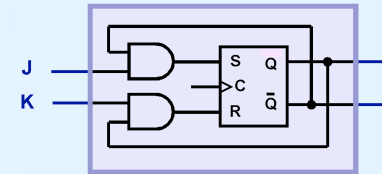
- If we can be sure that the inputs to an SR flip-flop will never both be 1, we will never have an unstable circuit. This may not always be the case.
- The SR flip-flop can be modified to provide a stable state when both inputs are 1.

- This modified flip-flop is called a **JK flip-flop**, shown at the right.
 - The “JK” is possibly in honor of Jack Kilby (inventor of the integrated circuit, 1958).



3.6 Sequential Circuits

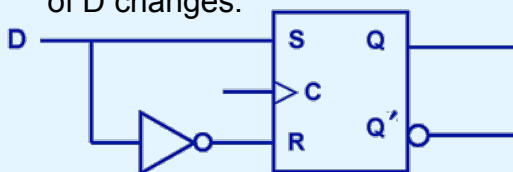
- At the right, we see how an SR flip-flop can be modified to create a JK flip-flop.
- The characteristic table indicates that the flip-flop is stable for all inputs.



J	K	Q (t+1)
0	0	Q (t) (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	Q (t)

3.6 Sequential Circuits

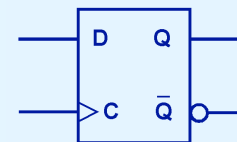
- Another modification of the SR flip-flop is the **D flip-flop**, shown below with its characteristic table.
- You will notice that the output of the flip-flop remains the same during subsequent clock pulses. The output changes only when the value of D changes.



D	Q (t+1)
0	0
1	1

3.6 Sequential Circuits

- The D flip-flop is the fundamental circuit of computer memory.
 - D flip-flops are usually illustrated using the block diagram shown below.
- The characteristic table for the D flip-flop is shown at the right.



D	Q (t+1)
0	0
1	1

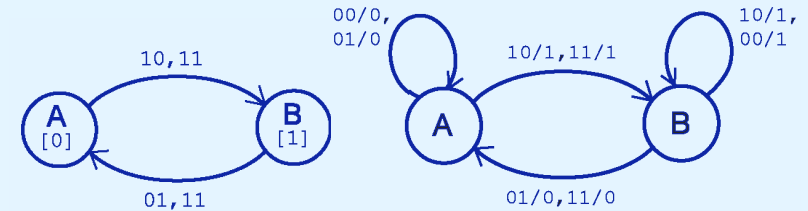
3.6 Sequential Circuits

- The behavior of sequential circuits can be expressed using characteristic tables or **finite state machines** (FSMs).
 - FSMs consist of a set of nodes that hold the states of the machine and a set of arcs that connect the states.
- Moore** and **Mealy** machines are two types of FSMs that are equivalent.
 - They differ only in how they express the outputs of the machine.
- Moore machines place outputs on each node, while Mealy machines present their outputs on the transitions.

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3.6 Sequential Circuits

- The behavior of a JK flip-flop is depicted below by a Moore machine (left) and a Mealy machine (right).

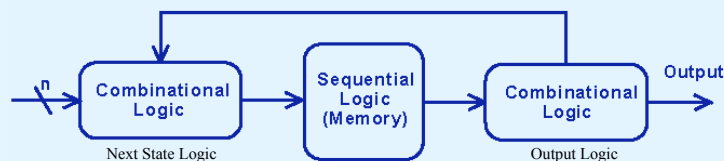
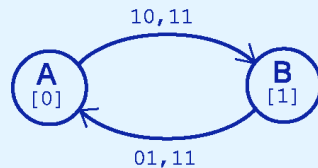


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3.6 Sequential Circuits

- Although the behavior of Moore and Mealy machines is identical, their implementations differ.

This is our Moore machine.

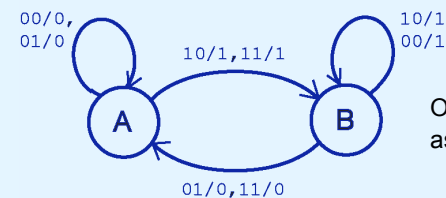
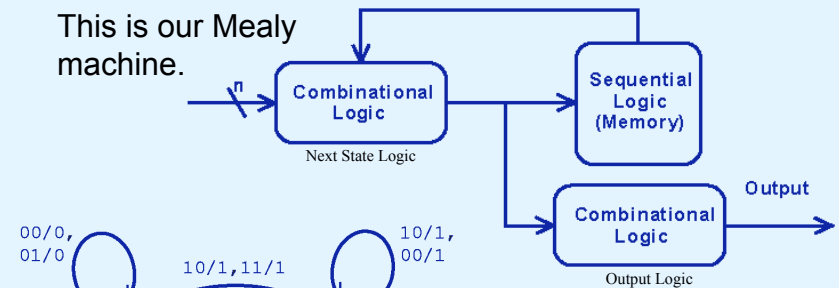


Output depends only on the current state.

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3.6 Sequential Circuits

This is our Mealy machine.



Output depends on the current state as well as the current input.

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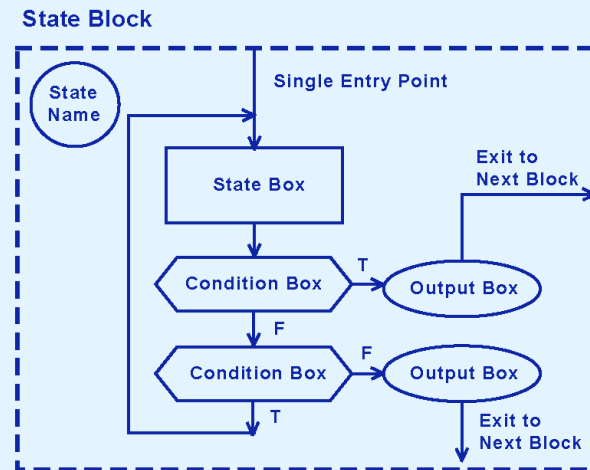
3.6 Sequential Circuits

- It is difficult to express the complexities of actual implementations using only Moore and Mealy machines.
 - For one thing, they do not address the intricacies of timing very well.
 - Secondly, it is often the case that an interaction of numerous signals is required to advance a machine from one state to the next.
- For these reasons, Christopher Clare invented the **algorithmic state machine** (ASM).

The next slide illustrates the components of an ASM.

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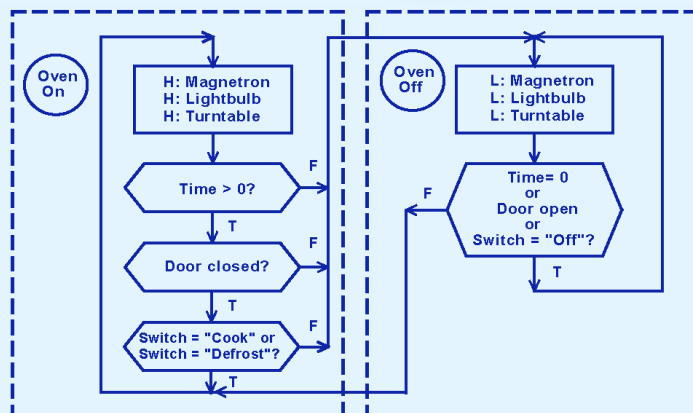
3.6 Sequential Circuits



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3.6 Sequential Circuits

- This is an ASM for a microwave oven.



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3.6 Sequential Circuits

- Sequential circuits are used anytime that we have a “stateful” application.
 - A stateful application is one where the next state of the machine depends on the current state of the machine and the input.
- A stateful application requires both combinational and sequential logic.
- The following slides provide several examples of circuits that fall into this category.

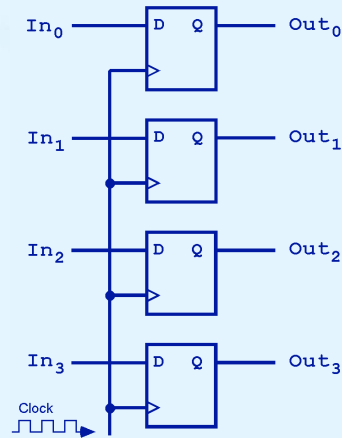
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3.6 Sequential Circuits

- This illustration shows a 4-bit register consisting of D flip-flops. You will usually see its block diagram (below) instead.

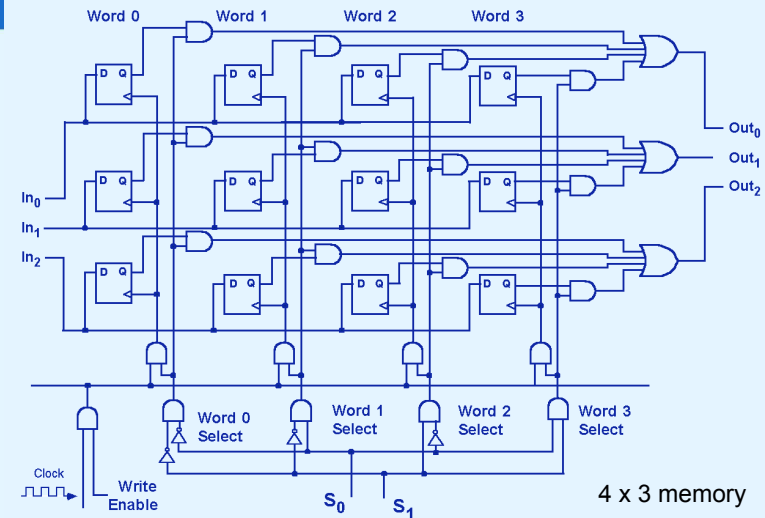


A larger memory configuration is shown on the next slide.



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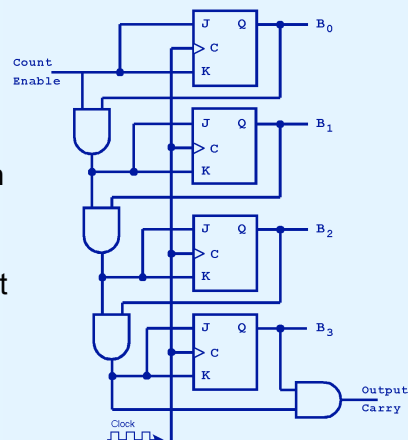
3.6 Sequential Circuits



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3.6 Sequential Circuits

- A binary counter is another example of a sequential circuit.
- The low-order bit is complemented at each clock pulse.
- Whenever it changes from 0 to 1, the next bit is complemented, and so on through the other flip-flops.



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3.7 Designing Circuits

- We have seen digital circuits from two points of view: digital analysis and digital synthesis.
 - Digital analysis* explores the relationship between a circuit's inputs and its outputs.
 - Digital synthesis* creates logic diagrams using the values specified in a truth table.
- Digital systems designers must also be mindful of the physical behaviors of circuits to include minute propagation delays that occur between the time when a circuit's inputs are energized and when the output is accurate and stable.

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3.7 Designing Circuits

- Digital designers rely on specialized software to create efficient circuits.
 - Thus, software is an enabler for the construction of better hardware.
- Of course, software is in reality a collection of algorithms that could just as well be implemented in hardware.
 - Recall the Principle of Equivalence of Hardware and Software.

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3.7 Designing Circuits

- When we need to implement a simple, specialized algorithm and its execution speed must be as fast as possible, a hardware solution is often preferred.
- This is the idea behind *embedded systems*, which are small special-purpose computers that we find in many everyday things.
- Embedded systems require special programming that demands an understanding of the operation of digital circuits, the basics of which you have learned in this chapter.

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Chapter 3 Conclusion

- Computers are implementations of Boolean logic.
- Boolean functions are completely described by truth tables.
- Logic gates are small circuits that implement Boolean operators.
- The basic gates are AND, OR, and NOT.
 - The XOR gate is very useful in parity checkers and adders.
- The “universal gates” are NOR, and NAND.

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Chapter 3 Conclusion

- Computer circuits consist of combinational logic circuits and sequential logic circuits.
- Combinational circuits produce outputs (almost) immediately when their inputs change.
- Sequential circuits require clocks to control their changes of state.
- The basic sequential circuit unit is the flip-flop: The behaviors of the SR, JK, and D flip-flops are the most important to know.

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Chapter 3 Conclusion



- The behavior of sequential circuits can be expressed using characteristic tables or through various finite state machines.
- Moore and Mealy machines are two finite state machines that model high-level circuit behavior.
- Algorithmic state machines are better than Moore and Mealy machines at expressing timing and complex signal interactions.
- Examples of sequential circuits include memory counters, and decoders.

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End of Chapter 3

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