Intermediaries between Abū Kāmil's and Fibonacci's algebras – lost but leaving indubitable traces

Jens Høyrup

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Abstract

It is an oft-repeated claim that Leonardo Fibonacci's algebra borrows from Abū Kāmil. A thorough analysis of the *Liber abbaci* as a whole and, in particular of course, of its chapter 15 part 3, the algebra, confirms this; but it also shows that at least many, plausibly all of the borrowings are indirect, and almost certainly that at least one group of borrowings go via a Latin intermediary which is not itself a direct translation of Abū Kāmil's algebra.

In order to see that one has, firstly, to know whether Fibonacci was faithful or creative when borrowing. Elsewhere in the *Liber abbaci* he demonstrates to be deliberately faithful. If that is taken into account in the analysis of his algebra confirms that a number of proble ms come from Abū Kāmil but not directly. Since, moreover, a number of the borrowings use the translation *avere* (a loanword from a Romance vernacular meaning "possession") for an initial non-algebraic unknown number (sometimes represented afterwards by a *res*, "thing" in the ensuing algebraic solution, sometimes by a *census*), the ultimate intermediary can be seen to have been Latin: there is no reason an Arabic treatise should borrow a Romance-vernacular term, and if against all odds this should have happened, then its orthography would hardly have survived the transcription into Arabic and Fibonacci's ensuing retranslation undistorted.

The terminological innovation demonstrates metamathematical acumen on part of the internediary: neither al-Khwārizmī nor Abū Kamil make this distinction between a non-algebraic and an algebraic *māl*.

Contents

To Barnabas Hughes
Friend of al-Khwārizmī and Fibonacci

Fibonacci's working style

In the following I shall analyze some of the borrowings in chapter 15 part 3 – the algebra – of Leonardo Fibonacci's *Liber abbaci*. In particular I intend to show that a number of borrowings from Abū Kāmil are indirect, and furthermore that at least many of them go via a lost Latin intermediary.

Basis for the argument is Fibonacci's characteristic working style as it is revealed by other parts of the *Liber abbaci*. A very clear example is the last problem of chapter 15 part 2 [B405;G622].[1]

The problem in question asks for three numbers (say, a, b and c) such that $^1/_2 a = {}^1/_3 b$, $^1/_4 b = {}^1/_5 c$, $a \cdot b \cdot c = a+b+c$. The solution proceeds by means of the false position $(a,b,c) = (8,12,15)$. This leads to $a+b+c = 35$, $a \cdot b \cdot c = 1440$. Therefore, the positions have to be reduced by a factor $\sqrt{^{35}/_{1440}} = \sqrt{^{7}/_{292}}$.

The *numerical* squares of a, b, and c are spoken of as tetragons (*tetragonus*). This Greek term is used regularly in the *Liber abbaci* about *geometric* squares (once, in a cistern problem, about a cube). Nowhere in the work except here does it refer to the square *of a number*. This probably means that Fibonacci here builds on a Greek, ultimately Byzantine source (which he may have encountered in Sicily as well as in Byzantium). It is not quite certain, however – a Latin translator from the Arabic *could* have used it for *murabbaʿ*.

A borrowing it is in any case. The way it is done illustrates how Fibonacci's deals with adopted material. The solution and discussion of the problem is divided in three sections, the second of which is an inserted generalizing explanation. In the first, the term *tetragonus* is used 11 times, while the synonym *quadratus* is absent. In the second, which explains why a square root has to be taken, *quadratus* is used 5 times, *tetragonus* never; this is obviously an explanation added by Fibonacci, in which he uses his own language. In the last section, which verifies the outcome and which can be presumed to continue the borrowing, *quadratus* is absent, while *tetragonus* is used 16 times.[2]

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[1] [B*m*;G*m*] refers to p. *m* in [Boncompagni 1857] and p. *n* in Enrico Giusti's new critical edition. Since the latter is likely to be found at present in only a few libraries while several good scans of the former are available at Google Books, these double references seem mandatory.

[2] This distribution is statistically extremely significant irrespective of the model we use. The simple model that the probability to choose *tetragonus* is $^{27}/_{32}$ and that to choose *quadratus* is $^{5}/_{32}$) shows the probability of the present distribution to be slightly below 10^{-6}. A model based on combinatorics

As we see, Fibonacci is highly faithful to the original when he borrows, but he does not emulate its style in added material or commentaries. He combines faithfulness with *deliberate* avoidance of imitation or pastiche – these would have induced him to carry over *tetragonus* to the commentary in the middle.

This and other instances of faithful copying are not evidence that Fibonacci was a compiler who did not understand what was in his book. The explanation in the middle section, and copious parallels, shows the contrary. Faithful copying was presumably a strategy making sure that no unintended misunderstandings would creep in. We may think of what Charles Homer Haskins [1924: 152] said about the 12th-century *de verbo ad verbum* translations of Greek texts (in part paraphrased from a 12th-century text). It did not reflect incompetence. Instead,

> Who was Aristippus that he should omit any of the sacred words of Plato? Better carry over a word like *didascalia* than run any chance of altering the meaning of Aristotle. Burgundio might even be in danger of heresy if he put anything of his own instead of the very words of Chrysostom.

Similar faithfulness can be seen in the lettering of diagrams, in the *Liber abbaci* as well as in other works from Fibonacci's hand. Diagrams that from the context can be seen to belong with Fibonacci's own explanations (mostly quite simple diagrams) are lettered *a-b-c-...*, while those that seem to belong with borrowings are lettered *a-b-g-...*[3]

An illustrative example is offered by the beginning of the *Pratica geometrie* [ed. Boncompagni 1862: 2–6]: At first comes a diagram proving *Elements* I.28 (not identified but following a generic reference to Euclid); it is lettered *a-b-g-d-e-...* and is almost certainly borrowed from the version translated directly from the Greek [ed. Busard 1987: 42]. Somewhat later, when Fibonacci explains why the area of a "quadrilateral and equiangular field" is the product of the sides, an illustrating diagram is lettered *a-b-c-d-e-f-g;* when going on with more complicated divisions of the square, Fibonacci returns to *a-b-g-...*.

An analogous case is found in the "Letter to Theorodus" [ed. Boncompagni 1862: 279–283]. At first the question asked by Theodorus, philosopher at Frederick II's court, is transformed by means of an *a-b-c-...* diagram; since this depends on the question that was asked, it must obviously have been invented by Fibonacci for the occasion. But the transformation allows Fibonacci to draw on already existing theory, and then all diagrams are lettered *a-b-g-...*.

(taking for granted that *tetragonus* occurs 27 and *quadratus* 5 times and supposing that the distribution is random) gives a probability of ${}^{5!\times27!}/_{32!}$, close to $5\cdot10^{-6}$.

[3] A few times, letter sequences later in the alphabet turn up. They do so when continuing an argument making use of the initial part of the alphabet; they thus are of no import.

Other examples from the *Liber quadratorum* and the *Liber abbaci* could be given, but this should suffice as illustrations.

Fibonacci's algebra

Let us therefore now turn to *Liber abbaci* chapter 15 part 3. It consists of two pieces. First (3A) the general rules, next (3B) 99 problems with interspersed theoretical observations.

3A corresponds to the general rules as presented by al-Khwārizmī, but with an important difference: Fibonacci has already dealt with the arithmetic of roots and binomials in chapter 14 and does not repeat. As revealed by Nobuo Miura [1981], there are traces in the text of Gerard of Cremona's translation of al-Khwārizmī, but no direct quotations. This, the *a*-*b*-*c*-... lettering of most diagrams, and the synthetic rather than analytic style of certain demonstrations are clear evidence that here Fibonacci is on his own. There is no reason to elaborate in the present context.

3B, a collection of 99 questions, is what I shall discuss.[4] It is an old observation that some of these share the mathematical structure with problems found in al-Khwārizmī's algebra, at times also the numerical parameters; similarly, some problems (with overlap with the former group) relate in one of these or both ways to Abū Kāmil's algebra. Some are also related to what can be found in al-Karajī's *Fakhrī*. This overlap already indicates that problems circulated widely, and that shared problems, even with shared parameters, do not prove any direct relationship between two texts (obvious ans well known, too often forgotten). In view of Fibonacci's tendency to be faithful to his sources we may further conclude that problems whose structure Fibonacci shares with either al-Khwārizmī or Abū Kāmil without sharing the parameters or the procedure can hardly have been borrowed directly from these predecessors.

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[4] The precise number depends on which variants are counted as independent questions. I follow the list in [Hughes 2004: 350–361], which along with [Boncompagni 1857] draws on the edition of chapter 15 in [Libri 1838: II, 307–479], based on a different manuscript of the *Liber abbaci*, and on Benedetto da Firenze's vernacular translation of the questions as rendered in [Salomone 1984]. A problem referred to as [H#*m*;G§*n*] is number *m* in Barnabas Hughes' list and §XV.*n* in [Giusti 2020]; [H#*n*] refers to #*n* in [Hughes 2004].

My reference for the comparison of Fibonacci's text with al-Khwārizmī will be Gerard of Cremona's Latin translation [ed. Hughes 1986]; it is indeed closer to the Arabic original than the extant Arabic texts, all later by a small century or more [Høyrup 1998; Rashed 2007: 83, 86]. When referring to the Arabic text I shall use [Rashed 2007]. For Abū Kāmil's text my reference will be Roshdi Rashed's edition.

The al-Khwārizmī cluster

In order to go beyond this negative conclusion we need to observe that Fibonacci's collection of problems in 3B contains a number of closed clusters, groups of problems that must have been adopted together.

The first eleven problems constitute an obvious cluster. Nine of them share the mathematical structure of a problem from al-Khwārizmī's algebra: five from his list of six illustrations of the basic cases, four from his collection of "various problems". Internally in each of these groups, they follow al-Khwārizmī's order, but the two groups are mixed up.[5] Of the nine that have a counterpart, that is, the same mathematical structure, only two share al-Khwārizmī's numerical parameters. Only one [H#10;G§243] has the same initial formulation as Gerard of Cremona's translation of al-Khwārizmī, which however is so simple that the coincidence might well be accidental; in that case, moreover, the numerical parameters differ, and so do the procedures.[6] Given his faithfulness when he copies, Fibonacci cannot have used al-Khwārizmī's *Algebra* (in Gerard's or any other version) *directly* for this sequence. There can be no doubt, however, that he drew on an introductory work descending from that model, compiled by a writer who was less faithful than Fibonacci when cherry-picking from a model.

A transformed problem from Abū Kāmil

A problem that illustrates Fibonacci's relationship to Abū Kāmil is [H#21;G§288], another "divided ten". In letter formalism:

$$10 = a+b \ , \ \ (^{a}/_{b}+10)\cdot(^{b}/_{a}+10) = 122\,^{2}/_{3} \ .$$

Abū Kāmil [ed. trans. Rashed 2012: 410*f*] solves the same problem; al-Karajī instead gives the sum as $143\,^{1}/_{2}$ [Woepcke 1852: 94]. It appears from Woepcke's paraphrase that al-Karajī posits a to be a *thing;* a simple transformation then reduces the problem to "*census* plus 16 made equal to 10 roots", one of the standard cases. Abū Kāmil posits $^{a}/_{b}$ to be a "large *thing*" (presupposing $a>b$), and $^{b}/_{a}$ to be a "small thing". Then, if R stands for the "large *thing*" and r for the "small *thing*",

[5] With Q referring to the six illustrating questions, V to the *varia*, and – indicating absence of a counterpart, Fibonacci's order is V1, –, Q2, Q3, –, Q4, Q5, V2, Q6, V4, V5. Using a simple combinatorial model we find that the odds that the order of borrowings from the two groups should be conserved by accident is $^{1}/_{4!\cdot 5!} = {}^{1}/_{2880}$.

[6] The problem, of type "divided ten" can be expressed

$$10 = a+b \ , \ \ ^{a}/_{b}+{}^{b}/_{a} = 3\,^{1}/_{3} \ .$$

We shall return to it.

$$(R+10)\cdot(r+10) = 122\tfrac{2}{3}\ ;$$

since $rR = 1$ we thus have

$$1+10\cdot(R+r)+100 = 122\tfrac{2}{3}\ ,$$

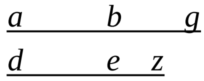
whence

$$R+r = 2\tfrac{1}{6}\ .$$

That is, the problem is reduced to

$$10 = a+b\ ,\quad {}^{a}/_{b}+{}^{b}/_{a} = 2\tfrac{1}{6}\ ,$$

which Abū Kāmil has already dealt with.



Fibonacci uses a line diagram, lettered *a-b-g-d-e-z*. Here, $ab = de = 10$, while $bg = {}^{a}/_{b}$, $ez = {}^{b}/_{a}$. Abū Kāmil's two algebraic unknowns are thus replaced by line segments. The procedure is parallel to that of Abū Kāmil, and also leads to the same reference to what has already been dealt with – actually however a reference to what has been dealt with by Fibonacci's source! Fibonacci himself [H#10,243;G§243] has treated the case where the sum of the two fractions is $3\tfrac{1}{3}$, not $2\tfrac{1}{6}$, as we have seen (above, note 6). A clear indication of copying – not directly from Abū Kāmil, however, but at most (and, in view of the shared structure of the argument, probably) from a source building on but reshaping Abū Kāmil's solution.

In the end Fibonacci says that the reader should know that

> when you have two numbers and divide the larger by the smaller and the smaller by the larger and multiply that which resulted from one division in that which resulted from the other, then from their multiplication always 1 is generated, and therefore I said 1 to come from *bg* in *ez*.

Once again we have a faithfully borrowed text (as revealed by the *a-b-g* lettering), with an added personal explanation which is separate from and not integrated in what was borrowed.

Al-Karajī, as we notice, offers a regular *al-jabr* solution. Abū Kāmil's reduction makes use of a technique that rather belongs with the *regula recta* with two unknowns (the problem to which he reduces the present one is then solved by means of *al-jabr*); Fibonacci, and his source, also remove anything that could make one think of *al-jabr* techniques (with the same proviso).

The *avere* cluster

This example shows Fibonacci's use of an intermediate source but nothing about where this source was produced. Such information can be derived instead from a cluster beginning at [H#62;G§387] and continuing at least until [H#95;G§675] (with interspersed additions from Fibonacci's own hand). It is characterized by using in all problems that ask for a single unknown number the term *avere* (borrowed from some Romance vernacular and meaning "possession") for this unknown, obviously a loan translation of *māl*. This term is used nowhere else in the *Liber abbaci*, and it is never used about the algebraic "second power", which even within this sequence is always *census*.[7] Sometimes, indeed, the *avere* is posited afterwards to be a *census*, sometimes to be a *thing*.

The *avere* must already have been present in Fibonacci's source. There is no reason that Fibonacci should suddenly on his own choose a new translation – earlier problems as well as the very last [H#99;G§682] use the standard translation *census* for *māl* in both roles; nor any reason that elsewhere but not here he should regularly replace an original Arabic initial *māl* by *numerus* or *quantitas*. We cannot exclude that this source was already written in a Romance vernacular (Italian, Catalan, Provençal or Castilian, though Italian seems even more unlikely than the others); much more plausible, however, is a Latin translation prepared in a Romance-speaking environment which borrows terms from its local vernacular (still Catalan, Provençal or Castilian, hardly any Italian vernacular); the translated text would likely have been created in al-Andalus. As we know from correct references to Abū Kāmil's algebra in the *Liber mahameleth*, Abū Kāmil's work circulated there.

Close to half of the problems in the cluster have a close counterpart in Abū Kāmil's algebra. Moreover, their order is as in that work. Enough to show that the compiler of the original drew on Abū Kāmil's text (perhaps indirectly), but also sufficiently few and scattered to prove that this was really an independent treatise and no mere redaction.

We may take a random but characteristic example [H#77;G§539], showing first how Abū Kāmil presents it:[8]

> If it is said, a *māl* of which the two roots and the root of its half and the root of its third are equal to it, how much is this *māl?*
>
> One solves it like this: posit your *māl* to be a *māl*. Then you say, two roots and the root of half of the *māl* and the root of a third of the *māl* are equal to a *māl*. The *thing* is thus equal to two and root of one half and root of one third, which is the root of the *māl*, and the *māl* is 4 and a half and a third and root of eight and root of five and a third and root of two-thirds.

In symbolic problematic translation – provocatively using x^2 for the *māl* and x for its

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[7] In those problems from the cluster that ask for a divided 10 or for two numbers it is not used.

[8] I translate from Rashed's French translation [2012: 440], checking with the Latin translation [ed. Sesiano 1993: 397] and – to my restricted ability – with certain key terms from the Arabic text.

root:

$$x+\sqrt{\frac{x^2}{2}}+\sqrt{\frac{x^2}{3}} = x^2 ,$$

which it is obvious for us to transform into

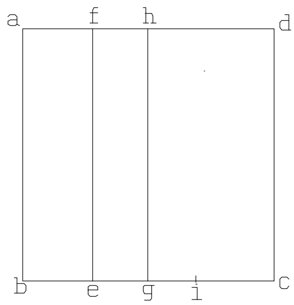
$$(2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}})x = x^2 ,$$

a clear case of “roots made equal to *census*”, whose rule just as our symbolic solution leads to

$$x = (2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}) .$$

As we can see from Abū Kāmil’s text he understands this to perfection. He is not allowed to say it, however: according to a canon that remained in force in Arabic as well as Latin (and post-Latin) algebra until the 16th century, irrational “coefficients” were unacceptable – see [Oaks 2017].[9] Therefore Abū Kāmil jumps directly to the result.

The *Liber abbaci* version of the problem [H#77;G§539] evades the difficulty by offering a geometric proof:[10]



> There is an *avere*, of which 2 roots and the root of its half and the root of its third are equal to it. Posit for this *avere* a *census*; and because two things and the root of the half of a *census* and the root of the third of a *census* are made equal to the *census*, make of the above-written square *ac* a *census*, and two roots of the same *census* will be the surface *dg*, and let the root of the half of the *census* be the surface *eh*, and the root of the third of the *census* the surface *bf*. Therefore *cg* will be 2 and *eg* will be the root of $\frac{1}{2}$ dragma, and *be* will be the root of the third of a dragma. And thus the whole *bc*, which is a *thing*, will be 2 and root of $\frac{1}{2}$ and root of $\frac{1}{3}$. Therefore multiply this in itself, and $4\frac{5}{6}$ and root of 8 and root of $5\frac{1}{3}$ and root of $\frac{2}{3}$ of one dragma results for the *census*, that is, the *avere* that was asked for. [followed by an explanation of the multiplication $(2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}})\cdot(2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}})$].

Two observations can be made. Firstly we see that Fibonacci considers it in need of no explanation that an area $\sqrt{(\frac{1}{2}}$ *census*) applied to a line of length 1 *thing* produces a breadth $\sqrt{\frac{1}{2}}$ (and similarly for $\frac{1}{3}$); that is, the tacit knowledge used by Abū Kāmil serves even here, no new theoretical insight intervenes.

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9 That it *was* a canon and not a consequence of failing ability to understand is obvious from how the taboo was circumvented by Abū Kāmil as well as Fibonacci.

For conceptual divergence, canons and taboos, cf. [Høyrup 2004].

10 The diagram belongs with the previous problem but is recycled.

Secondly we notice the lettering *a-b-c-...* . There are a few diagrams lettered *a-b-g-...* within the *avere* cluster, but the large majority are of type *a-b-c-...*, in contrast to what we find in the preceding sections of 3B. So, even though the appearance of the *avere* shows that Fibonacci drew on an existing Latin (or possibly Romance-vernacular) reinterpretation of Abū Kāmil's algebra, he seems to have intervened himself – unless the Latinizing letter sequences should belong to his source, which would go against the habits of extant Latin mathematical translations from the Arabic.

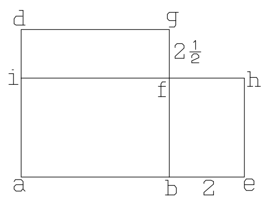
Inspection of the very first problem of the cluster [H#62;G§387] may show us what happened. The preceding problem [H#61;G§383] builds on a diagram lettered *b-g-d-e-f-h* (*a* is absent because the corresponding corner of the square is not spoken of and therefore not marked at least in thew *Liber abbaci*). Fibonacci borrows from that problem a correct but redundant reference to the classes of Euclidean binomials, which are only mentioned in these two problems of chapter 15 part 3 and the one coming just before. Together with the idea to refer to the Euclidean classes he appears to have taken over the diagram but then adapted it to his own purpose – namely for solving a problem that he is not allowed to speak of as $(10+\sqrt{30})$*things*$+20$ = *census* (now lettered *a-b-d-e-f-h-i-c* – this time *d* is left out because the corresponding corner is not spoken about).

Personal intervention also seems to be indicated by a number of phrases similar to those that are used elsewhere when an extra explanation or a supplementary proof are provided – *ad hoc itaque demonstrandum* [H#62;G§389], *quod ostendam in figura* [H#63; G§394], *quod per figuram geometricam demonstrare curavi. Ponam* [H#64;G§403], *et nos ponamus hec in figura, ut que dicere volumus clarius videantur* [H#71;G§456], etc. While we cannot know whether already Fibonacci's source for the *avere* cluster used geometry to evade the tabooed irrational coefficients, the way it is done appears to point to Fibonacci himself.

The money-sharing cluster

A last cluster of interest contains problems about a constant or varied amount of money shared between different numbers of men. The first of them [H#12;G§252] runs:

> I divided 60 between some men, and something resulted for each; and I added two men above them, and between all these I divided 60, and for each resulted $2\frac{1}{2}$ less than resulted at first. Let the number of the first men be the line *ab*, and on it is erected at a right angle the line *bg*, which should be that which falls to each of them of the mentioned 60 *denarii*, and draw the line *gd* equal and parallel to the line *ba*, and the straight line *da* is connected. Then the space of the quadrangle *abgd* will be 60, when it is connected [*colligatur*] by *ab* in *gb*. Then protract the line *ab* to the point



e, and let *be* be 2, that is, the number of men to be added. And on the line *bg* the point *f* is marked, and let *gf* be $2\frac{1}{2}$, that is, that which each one got less by the addition of two men. And through the point *f* the line *hi* is protracted equal and parallel to the line *ea*, and the straight line *eh* is connected; the quadrangle *heai* will be 60, since it is contained by *ae* in *eh*, namely by *ae* in *bf*, where *bf* is that which resulted for each of the men *ae* from the 60 *denarii*. The surface *ei* is thus made equal to the surface *bd*. The multiplication of *gb* in *ba* is thus made equal to the multiplication of *ea* in *fb*. Whence these four lines are proportional. Therefore, the first *gb* is to the second *fb* as the third *ea* to the fourth *ba*, whence, by dividing,[11] as *gf* is to *fb*, so is *fb* to *ba*. But the ratio *gf* to *eb* is as 5 to 4. Thus *fb* contains once and one fourth the number *ba*.

So, posit for the number *ab* a *thing*. *bf* will thus be $1\frac{1}{4}$ *thing;* and multiply *ab* in *bf*, and $1\frac{1}{4}$ *census* results for the surface *bi* [...].

There are no problems similar to this in the original text of Al-Khwārizmī algebra as reflected in Gerard of Cremona's translation nor in the collection of supplementary problems which he "found in another book" (that is, in another manuscript version containing 21 of the problems that later were adopted into the "living" Arabic text); the type is equally absent from Robert of Chester's somewhat expanded version [ed. Hughes 1989]. In the Arabic manuscripts (all later than the two translations[12]) we have a variant where the amount to be distributed is 1 dirham and one man is added, which results in a difference of $\frac{1}{6}$ [ed. trans. Rashed 2007: 190*f*]. This may have crept into the tradition at any moment before 1222; there is no reason to believe it inspired Fibonacci, neither directly nor indirectly.

Instead, Fibonacci's problem is close to a similar one proposed by Abū Kāmil [ed. trans. Rashed 2012: 352–355]. Here, 50 dirham are shared first among some men, them among 3 more, the difference between what each one gets in the two situations being $3\frac{3}{4}$ dirhams. The solution follows the same pattern as that of Fibonacci, but instead of using proportions the argument about the diagram is arithmetical all the way through.

In the next problem in the *Liber abbaci* [H#13;G§259], first 20 is divided between some number of men, next 30 between 3 more; the difference between the shares in the two situations is 4. Here, a slightly more complicated diagram is used, lettered *a-b-g-d-e-...*; proportion techniques are used again, and followed by an algebraic solution of the resulting equation). Abū Kāmil offers four problem of the same structure [ed. trans. Rashed 2012: 358–371], presenting solutions based on kindred diagrams and never

[11] That is, we transform $\frac{gb}{fb} = \frac{ea}{ba}$ into $\frac{gb-fb}{fb} = \frac{ea-ba}{ba}$, from which follows that $\frac{gf}{fb} = \frac{eb}{ba}$. That could, by the way, be seen directly in the diagram, just by removal of the shared surface *af* from both of the surfaces *ag* and *ah*. "Permutation" leads to $\frac{gf}{eb} = \frac{fb}{ba}$.

[12] The oldest is dated 1222 [Rashed 2007: 85].

referring to proportions.

The following problems in the *Liber abbaci* [H#14–15;G§266,271] have the same mathematical structure. The solutions, however, are based on diagrams lettered *a-b-c-d-e-...*; the algebraic *thing* and *census* enter directly in the discussion of the diagrams, while proportions go unmentioned. The changing lettering of the diagrams suggest that the solutions to [H#12–13] build on a source that ultimately goes back to Abū Kāmil while those of [H#1415] are of Fibonacci's own brew. The former reformulate Abū Kāmil's solution in proportion terms; Fibonacci, in his own (more straightforward) solutions, does not mention them.

In the last problem from the sequence [H#16;G§276], first 10 are divided between a certain number of men, next 40 between 6 more; they get the same in the two cases. Thinking in terms of proportions would lead to $\frac{h+6}{h} = \frac{40}{10}$, and thus $\frac{6}{h} = \frac{40-10}{10}$, and finally $6 \cdot 10 = 30 \cdot h$. But Fibonacci, again apparently working on his own, has no such preferences on the present occasion. He observes, without any appeal to algebra, that the 30 extra monetary units must be the share of the 6 extra men, each of whom therefore gets 5. Since the first men get the same, their number must be $10 \div 5 = 2$.[13]

There can be no doubt that the sequence [H#12–16] is part of a cluster adopted from a single source, which on its part seems to have been inspired by Abū Kāmil (though shared inspiration cannot be excluded); for the last three problems, however, Fibonacci seems to have presented a simpler solution of his own making). Since [H#11] belongs to the cluster borrowed indirectly from al-Khwārizmī's algebra, [H#12] must be the first member of the present cluster. Whether it extends beyond [H#16] seems undecidable but rather unlikely according to internal criteria of style.

There is no direct evidence for where the original was produced from which Fibonacci took this cluster, not even for whether he used an Arabic text or a Latin translation. At most we can say that the predilection for proportion techniques may make us think of the *Liber mahameleth*, which again would point to al-Andalus.

[13] The problem is also in Abū Kāmil's algebra [ed. trans. Rashed 2012: 370–373]. First Abū Kāmil gives two unexplained numerical prescription, the second of which coincides with the one given by Fibonacci, stating afterwards that "the reason of that is obvious"; next he actually formulates the proportion $\frac{h}{h+6} = \frac{10}{40}$, identifying then the second ratio with the number $^{1}/_{4}$. Positing h to be a *thing* he gets an algebraic equation.

A concluding remark

We can thus discern two distinct clusters in Fibonacci's collection of algebraic problems, well apart in his text, both of which draw on Abū Kāmil's algebra but do so indirectly. Clear differences in mathematical style indicate that the two clusters derive from different intermediaries; for one of them at least Fibonacci relies on a Latin translation.

My feeling is that both intermediaries were produced in al-Andalus, probably during the 12th century; this, if true, would make it rather unlikely that they should turn up in Arabic manuscript libraries. But feelings may be mistaken, so it *could* still happen. As to the Latin model for the *avere* cluster, we should remember that the *Liber mahameleth* is contained in a manuscript that had been studied by Michel Chasles as well as Louis Karpinski but still had to wait for Jacques Sesiano before it was discovered; a similar miracle can still be hoped for.

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