A diluted al-Karajī in Abbacus Mathematics

Jens Høyrup
jensh@ruc.dk
http://akira.ruc.dk/~jensh

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In several preceding Maghreb *colloques* I have argued, from varying perspectives, that the algebra of the Italian abbacus school was inspired neither from Latin algebraic writings (the translations of al-Khwārizmī and the *Liber abbaci*) nor directly from authors like al-Khwārizmī, Abū Kāmil and al-Karajī; instead, its root in the Arabic world is a level of algebra (probably coupled to *muᶜāmalāt* mathematics) which until now has not been scrutinized systematically.

Going beyond this negative characterization I shall argue on the present occasion that abbacus algebra received indirect inspiration from al-Karajī. As it will turn out, however, this inspiration is consistently strongly diluted, and certainly indirect.

1. Al-Khwārizmī, Abū Kāmil and al-Karajī

Let us briefly summarize the relevant aspects of what distinguishes al-Karajī from his algebraic predecessors.

Firstly, there is the sequence of algebraic powers. Al-Khwārizmī [ed., trans. Rashed 2007], as is well known, deals with three powers only: *census* (to adopt the translation which will fit our coming discussion of abbacus algebra), roots, and simple numbers. So do ibn Turk [ed., trans. Sayılı 1962] and Thābit ibn Qurrah [ed., trans. Luckey 1941] in their presentation of proofs for the basic mixed cases, which indeed involve only these same powers.

Abū Kāmil [ed. Sesiano 1993: 398] adds to these *cubus*, *census census* and *cubus cubi*, explaining them respectively as the product of *thing* (not root) and *census*, of *census* and *census*, and of *cubus* and *cubus*; later (*ibid.* p. 404) *census census census* (the same as *cubus cubi*, as observed by Abū Kāmil) and *census census census census* turn up without being explained.

In the *Kāfī* [ed., trans. Hochheim 1878], al-Karajī stays with the original three powers; so do a number of later elementary presentations of *al-jabr*. I shall therefore say nothing more about these works. The *Fakhrī*, in contrast, starts by presenting the endless sequence of ascending powers and their reciprocals [Woepcke 1853: 48*f*], while the *Badī*ʿ makes use of them on the base of a reference to the *Fakhrī* [trans. Hebeisen 2008: I, 108].

Next, there is polynomial calculus. Al-Khwārizmī [ed., trans. Rashed 2007: 122–133] teaches how to multiply monomials and binomials of no more than the first degree, and how to add and subtract polynomials of no more than the second degree; since he has no negative powers and no power beyond the *census*, this exhausts what can be asked for within his framework. Though having no negative powers, he is able to divide by a first-degree polynomial and to get rid of the division as the first step in the reduction of an equation – the first of the mixed problems where it happens [ed., trans. Rashed 2007: 164–167] can for convenience be summarized as

$$\frac{2\tfrac{1}{2}\ thing}{10-thing} + 5\,thing = 50\ ,$$

which al-Khwārizmī immediately translates into

$$\frac{2\tfrac{1}{2}\ thing}{10-thing} = 50 - 5\,thing\ .$$

Therefore, since "when you multiply that which results from a division by that which is divided, it gives back your possession",[1]

$$2\tfrac{1}{2}\ thing = (10-thing)\cdot(50 - 5\,thing)\ .$$

But this is as far as he manages to go. A previous problem [ed., trans. Rashed 2007: 162*f*] with the structure

$$\frac{thing}{10-thing} + \frac{10-thing}{thing} = 2\frac{1}{6}$$

is replaced immediately and without argument with

$$thing^2 + (10-thing)^2 = thing\cdot(10-thing)\cdot 2\frac{1}{6}\ .$$

Possibly, al-Khwārizmī could mentally deal with two polynomial divisors at a time (but his hidden argument could also have been geometric, as it can be seen

[1] My translation into English, as all such translations in what follows.

to be elsewhere in a similar case). However, the algebraic apparatus he expounds does not allow him to convey the argument to the reader.[2]

As said above, al-Khwārizmī shows the addition and subtraction of polynomials until the second degree [ed., trans. Rashed 2007: 136–143]. For binomials a geometric two-axis argument is given, for trinomials the corresponding figure is stated to be incomprehensible, while the argument from words is compelling.

What Abū Kāmil does on this account in the introduction is not very different.[3] He explains the same three powers, gives some new geometric arguments for the multiplication of polynomials, and on the other hand omits the explicit explanation of the additive and subtractive operations. The elimination of a division by an algebraic expression is explained within the problem section as done by al-Khwārizmī (and in the context of same problem) [ed. Sesiano 1993: 360*f*], and the elimination of two divisions at a time once again goes unexplained (and again in the same problem) [ed. Sesiano 1993: 365]. When the problem introducing *cubus, census census* and *cubus cubi* ([ed. Sesiano 1993: 398*f*], cf. above) has been reduced to

$$1 \textit{ census census} + 12^1/_4 \textit{ census} = 9 \textit{ cubi} ,$$

Abū Kāmil asks to "reduce everything to 1 *census*", using the same expression as when an equation is normalized through division by the coefficient of the highest-degree member, and giving no further explanation. Even though he has not introduced the topic before, Abū Kāmil thus sees division of higher powers by a *census* as something trivially familiar. If this was a new technique developed by Abū Kāmil himself, he would certainly have explained it, and probably pointed out his own role; we may conclude that it was not.

In the *Fakhrī*, al-Karajī explains that the sequences of ascending powers and their reciprocals form two geometric series starting from unity, and shows how to divide a polynomial by a power [Woepcke 1853: 53]; formulates the general rules for the addition and subtraction of polynomials (ibid., pp. 55*f*); and points out that three-term equations in general can be solved in the same way as second-

[2] Both of these problems are also in Gherardo of Cremona's translation [ed. Hughes 1986: 251*f*]. Even though some of the mixed problems are likely to have crept in later, there is thus no reason to doubt that these two belong to al-Khwārizmī's original stock.

[3] No good edition of a good manuscript exists. I have consulted [Levey 1966], [Sesiano 1993] and [Chalhoub 2004]. However, the deficiencies of manuscripts and / or editions should play no role for the present argument. My references will be to [Sesiano 1993], an edition of the certainly far from perfect Latin translation, which however presents the advantage to point out where it differs from the Arabic manuscript.

degree equations if one of the powers involved is the mean proportional between the other two (*ibid.* pp. 71*f*). He also (*ibid.* p. 63) states the theorem which in symbolic writing becomes

$$(\frac{a}{b}+\frac{b}{a})\cdot a\cdot b \;=\; a^2+b^2\ ,$$

which both al-Khwārizmī and Abū Kāmil use but do not enounce explicitly.

In the *Badīʿ*, the whole of book II [trans. Hebeisen 2008: I, 105–137] is dedicated to the extraction of roots of a polynomial – which evidently presupposes everything developed in the *Fakhrī* concerning the algebraic powers and the arithmetic of polynomials, and which goes beyond it when it comes to the division with a polynomial and in its more explicit use of the notion of degree. Polynomial arithmetic, though less explicitly and with more restricted scope, also underlies much of book III on indeterminate analysis [trans. Hebeisen 2008: I, 138–187].

Thirdly, we may look at how radicals and polynomials consisting of number and radical(s) are dealt with.

Already al-Khwārizmī was (at least practically) aware that polynomials containing radicals behave like algebraic polynomials under additive and subtractive operations[4] – he treats the two together, and goes directly from the statement (and later, the proof) that

$$(20-\sqrt{200}\,)-(\sqrt{200}\,-10)) \;=\; 10-2\cdot\sqrt{200}$$

to the statement respectively the proof that

$$(50+10\,\mathit{roots}-2\,\mathit{census})+(100+\mathit{census}-20\,\mathit{roots}) \;=\; 150-10\,\mathit{roots}-\mathit{census}$$

[ed., trans. Rashed 2007: 130*f*, 138–143]. He also gives (explicitly paradigmatic) examples illustrating the rules $n\sqrt{a} \;=\; \sqrt{(n\cdot n)a}$ and $\frac{1}{n}\sqrt{a} \;=\; \sqrt{(\frac{1}{n}\cdot\frac{1}{n})a}$ as well as $\frac{\sqrt{p}}{\sqrt{q}} \;=\; \sqrt{\frac{p}{q}}$, $\frac{a\sqrt{p}}{\sqrt{q}} \;=\; \sqrt{\frac{(a\cdot a)p}{q}}$, $\sqrt{p}\cdot\sqrt{q} \;=\; \sqrt{p\cdot q}$ and $\sqrt{\frac{1}{p}}\cdot\sqrt{\frac{1}{q}} \;=\; \sqrt{\frac{1}{p}\cdot\frac{1}{q}}$ [ed. Rashed 2007: 130–137].

Abū Kāmil [ed. Sesiano 1993: 349–355] does much the same, in part formulating things in general terms, and basing himself on geometric arguments.

[4] We may safely assume that the linguistic coincidence – both the first-degree power and the radical being a "root" – has facilitated this insight, which then ran into no trouble in calculational practice. Later, as book X of the *Elements* was assimilated, theoretical reasons would enforce the point – but since this did not affect abbacus algebra, there is no reason to elaborate.

However, he also observes [ed. Sesiano 1993: 355–358] that expressions $\sqrt{a} \pm \sqrt{b}$ can be simplified if *ab* (or $^{a}/_{b}$) is a perfect square;[5] for instance, $\sqrt{4} + \sqrt{9} = \sqrt{4+9+2\sqrt{4\cdot 9}} = \sqrt{25} = 5$, while $\sqrt{8} + \sqrt{18} = \sqrt{8+18+2\sqrt{8\cdot 18}} = \sqrt{50}$. For numbers not fulfilling the conditions he makes the terse observation [ed. Sesiano 1993: 358] that “the question is better than the answer” – namely, that the transformation of $\sqrt{10} + \sqrt{2}$ into $\sqrt{12+\sqrt{80}}$ is of no help (Abū Kāmil’s example).

This last innovation, just as Abū Kāmil’s occasional use of higher powers, may perhaps be seen as a starting point for some of al-Karajī’s more radical innovations, and probably as evidence that Diophantos was not the only inspiration for these. At first, however, the *Fakhrī* generalizes al-Khwārizmī’s rules for multiplying and dividing square roots with each other or with numbers to cube and quartic roots [Woepcke 1853: 56*f*]. He then (*ibid.* pp. 57*f*) goes on with the addition and subtraction of square roots, formulating the rules for when they are useful more clearly than Abū Kāmil, and with similar rules (and restrictions) for the addition and subtraction of cube roots, proving them by means of the development of $(a\pm b)^3$ (in the paradigmatic case $a = 3$, $b = 2$).)

The *Badīʿ* goes well beyond that. When transferring the theory of irrational magnitudes of *Elements* X to an extended realm of numbers, al-Karajī adds in the end an observation about the uncountable other bi- and polynomials similar to $\sqrt{10} + \sqrt[3]{15}$ and $\sqrt{10} + \sqrt[3]{15} + \sqrt[4]{20}$ [trans. Hebeisen 2008: 70]; afterwards he gives the same rules as the *Fakhrī* for multiplying and dividing monomials (*ibid.* pp. 76–79) and for adding and subtracting binomials, not however going until quartic roots (*ibid.* pp. 80–84). Then he goes on with the multiplication of non-algebraic polynomials (*ibid.* pp. 86–89); with the division by quadratic and even quartic binomials (*ibid.* pp. 90–94), giving up in front of trinomials; with the extraction of the square root of bi- and polynomials (*ibid.* pp. 95–102); and with the cubes of the Euclidean classes of irrationals.

[5] He also mentions the possibility that both *a* and *b* are perfect squares, without pointing out that this is a stronger condition.

2. Abbacus algebras

Considerations similar to several of al-Karajī's innovations turn up in Chapter 14 of Fibonacci's *Liber abbaci* [ed. Boncompagni 1857: 352–387]. However, since abbacus algebra did not take its inspiration from that book,[6] I shall not engage in analysis of similarities and differences, but instead turn to the abbacus treatises.

From the earliest beginning – namely the algebra chapter in Jacopo da Firenze's *Tractatus algorismi*[7] – abbacus algebra dealt not only with al-Khwārizmī's six fundamental cases (the equation types of the first and the second degree) but also with those that can be obtained from them through multiplication by a *thing* or a *census*, and with the biquadratic obtained from the fourth case through the substitution (*thing*, *census*) -> (*census*, *census census*). It thus makes use of the third as well as the fourth power of the unknown and gives correct rules for all these cases – just as al-Karajī had taught, but without stating the arguments as he had done. Very soon, certain abbacus masters also invented (false) rules for solving cubic and quartic equations that cannot be solved in this way. Our earliest source for this phenomenon is Paolo Gherardi's treatise from 1328.[8] However, Gherardi (who does not go beyond the third degree) does not

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[6] This is not the place to argue for this claim, but see my [2007].

[7] [Ed. Høyrup 2007]. In principle, this algebra chapter need not belong with Jacopo's original treatise from 1307 – in any work where we do not possess the manuscript made by the author anything can in principle have been added or changed between the preparation of the original and the writing of the shared archetype for existing manuscripts or editions. But even if this algebra should be a secondary insertion it still belongs to the early fourteenth century, predating all other known abbacus writings on algebra (the manuscript copy can be dated by watermarks to c. 1450); since it is obviously taken over wholesale (presumably translated from a Catalan or a Provençal source), it is uninteresting for anything but a biography of the otherwise unidentified Jacopo whether he or a near-contemporary of whom we do not even know the name carries responsibility for its adoption.

Van Egmond's attempt [2009] to date this algebra after 1390 builds on failing willingness or inability to read not only the sources (he only refers to equation types, never to the actual equations or examples nor their words, perhaps simply trusting old notes of his) but even his own past publications. See my forthcoming reply in the same journal.

[8] Known from a later copy, of which [Arrighi 1987] contains the complete text and [Van Egmond 1978] an edition of the algebraic section together with an English translation and a mathematical commentary.

Gherardi's false rules emulate those for the second degree blindly, suggesting that neither he nor their inventor understood why the rules for the reducible third- and fourth-present us with the full range of these fanciful inventions: a list of cases put together by Giovanni di Davizzo in 1339 (borrowed into the manuscript Vatican, Vat. lat. 10488 from 1424) is so close to Jacopo yet different on certain characteristic point that it can be seen to descend from a close source of Jacopo but not from the "Jacopo-algebra" itself – but it also contains one rule which is almost illegible[9] but which is certainly false and is certainly not dealt with by Gherardi. Giovanni di Davizzo is also interesting for what precedes the list of algebraic cases: first a sequence of rules for the multiplication of algebraic powers, next misshaped rules for the division of a lower by a higher power, where negative powers are identified with roots;[10] and finally arithmetic of (numerical, not algebraic) binomials, going (now correctly) until the reduction of $\sqrt{18} \pm \sqrt{8}$ and the determination of $\frac{35}{\sqrt{4}+\sqrt{9}}$ ($\sqrt{4}$ and $\sqrt{9}$ being treated as if they were irrational).

The full gamut of powers until the eighth is mastered by Dardi of Pisa in his *Aliabraa argibra* from 1344,[11] who also explains how to reduce higher-order equations through division [ed. Franci 2001: 78*f*]. Apart from 194 "regular" cases,[12] Dardi treats of four "irregular" cases of the third and fourth degree, cases whose rules are only valid under specific circumstances (circumstances not specified by Dardi). The dress of two of them is so different from what Dardi does elsewhere that we may be confident Dardi did not invent the group; on the other hand, their inventor, though cheating by pretending the rules are generally valid,[13] must have been quite competent in polynomial algebra.

Dardi's treatise also contains a long section about the arithmetic of numerical binomials, mostly consisting of examples in great number but also with more

degree cases work.

[9] Somebody discovered it was wrong and glued a piece of paper over it. This scrap has disappeared, but the glue has darkened the paper, making most of the writing illegible.

[10] See [Høyrup 2009: 56–59].

[11] [Franci 2001] contains an edition of one of the three main manuscripts (Siena, Biblioteca Comunale, I.VII.17). The other two (Vatican Library, Chigi M.VIII.170, Arizona State University Tempe) do not differ from the one published by Raffaella Franci in respects which concern the present discussion.

[12] Dardi reaches this impressing number by making ample use of radicals, *viz* of square and cube roots of numbers as well as algebraic powers).

[13] Indeed, nobody else of the many who copy them explains their restricted validity).

theoretical observations. Most place is taken up by the multiplication of second-degree monomials and polynomials, but we also find a multiplication of a cubic with a quartic root [ed. Franci 2001: 51] and examples of the addition and subtraction of square roots. In contrast to Giovanni di Davizzo, Dardi explains the condition under which the reduction is possible (*ibid.* p. 53]. He even explains (*ibid.* p. 59) how to divide by a binomial (the first example being $\frac{8}{3+\sqrt{4}}$; here, Dardi gives an argument which is likely to be his own invention and in any case not a borrowing from a tradition going back al-Karajī: it is based on the rule of three (the presence of a similar division in the Giovanni fragment suggests that the trick itself comes from tradition).

Toward the end of the fourteenth century, we find explicit expression of the idea that the ascending algebraic powers constitute a geometric series, namely in the algebra section of the manuscript Florence, Bibl. Naz., Fond. Princ. II. V. 152 [ed. Franci & Pancanti 1988], which on one hand contains some suggestions of fresh but indirect connections to the Arabic world,[14] but which on the other hand begins to replace the Arabic multiplicative naming of powers by naming according to the embedding principle (without doing so consistently). Whether the idea of the geometric progression is an independent observation or a borrowing is thus quite unclear.

Interest in the sequence of inverse power is documented in three encyclopedic abacus treatises from around 1460–70, one of which is Benedetto da Firenze's *Trattato de praticha d'arismetrica*.[15] All three depend on Antonio de' Mazzinghi's work in various respects (taking over, so it appears, even some of his marginal annotations), and it is a fair guess that the (indubitably existing) shared source for their successful treatment of negative powers and the ratio between powers

[14] *Censo* is used in one problem about a sum of money without the author understanding to the full (when having found this *censo* he feels obliged to find its root, only having to square it afterwards); and a scheme for the multiplication of trinomials which is very similar to what was made in the Maghreb.

[15] No complete edition exists, but several chapters from the manuscript Siena, Biblioteca Comunale degli Intronati, L.IV.21 have been published. [Arrighi 2004 / 1965] is a thorough description of the complete manuscript. As it turns out at closer analysis, this manuscript is Benedetto's working original (sometimes extensive marginal calculations were made before the main text was written).

The other two encyclopaediae are Vatican, Ottobon. lat. 3307 and Florence, Bibl. Naz. Centr., Palat. 573. Both can be seen in the same way to be their respective authors' working originals, and no copy of either is known.

should also be identified with Antonio – thus going back to the later fourteenth century. (Antonio, being the first to construct tables of composite interest, understood geometric progressions to the full.)

If we count the treatment of powers in Benedetto's and the two similar treatises as a reflection of what had been done by Antonio, I know of nothing in the fifteenth century abbacus record which elucidates our question – from 1400 onward, the Italian development can be considered fully autonomous. What we know from the fourteenth century, however, shows that abbacus algebra took over ideas, problems and results which do not come from al-Khwārizmī nor from Abū Kāmil, but which are all present in al-Karajī's advanced writings. On the other hand, what is reflected in the abbacus tradition is only the elementary aspect of al-Karajī's innovations (neither root extraction of polynomials nor division by a polynomial turns up in any abbacus source I know of). Though thus belonging to what can be characterized as an "al-Karajī tradition", it has nothing to do with the tradition which was carried by al-Samawʾal and his like. What is reflected in the abbacus tradition is *a diluted al-Karajī-tradition* – a perfect example of *Gesunkenes Kulturgut*.

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