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Influences of Institutionalized Mathematics Teaching on the Development and Organization of Mathematical Thought in the Pre-Modern Period. Investigations in an Aspekt of the Anthropology of Mathematics

by

Jens Høyrup

Zum Zusammenhang von Wissenschaft und Bildung am Beispiel des Mathematikers und Lehrbuchautors Martin Ohm

von

Bernd Bekemeier

Vorbemerkung

Der vorliegende Band ist im Rahmen des Projekts "Zum Verhältnis von Wissenschafts- und Bildungsprozeß - dargestellt am Beispiel der Entwicklung der Mathematik im 19. Jahrhundert" - entstanden. Dieses Projekt wird im IDM durchgeführt und von der Stiftung Volkswagenwerk finanziert und gefördert.

Wir messen Kooperationsbeziehungen zu Wissenschaftlern der verschiedensten Disziplinen, die an der Erforschung des Zusammenhangs von Wissenschaft und Bildung interessiert sind, für die Entwicklungen unserer eigenen Vorstellung und Konzepte eine große Bedeutung bei. Daher haben wir uns sehr gefreut, daß unser Kollege Jens Høyrup von der Universität Roskilde bereit gewesen ist, zu einem Vortrag an das IDM zu kommen, um auf der Grundlage eines Vergleichs verschiedener Perioden der Mathematikgeschichte einige Überlegungen zum Einfluß institutionalisierter Mathematikausbildung auf Entwicklung und Organisation des mathematischen Denkens vorzutragen. Wir möchten ihm an dieser Stelle ganz herzlich danken, daß er diese Überlegungen in einer sehr materialreichen Studie niedergeschrieben und uns zum Abdruck überlassen hat.

Die Fallstudie von Bernd Bekemeier über den Mathematiker und Lehrbuchautor Martin Ohm ist eine Vorarbeit zu einer umfangreicheren Untersuchung, die Zusammenhänge zwischen dem pädagogischen Denken des frühen 19. Jahrhunderts und der Herausbildung bestimmter metamathematischer Konzepte zum Gegenstand hat. Für die Erstellung des Manuskripts sagen wir Maria Ahrend, Pauline Haslam, Ingrid Kootz, Emmy Meyer und Hannelore Schoofs unseren herzlichen Dank.

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INHALTSVERZEICHNIS

ERRATA

Because the proof-sheets could not be submitted to Jens Høyrup, a number of errors have crept into the paper on "Influences of Institutionalized Mathematics Teaching ...". Those which interfere with the meaning of the text are listed below. Those which do not are left to the care of the reader.

p. 9 l.9 /of the historically/, read /of cognitive substance as present in the sociology of science. "Sociology of mathematical knowledge" would suggest both neglect of the historically/

p. 9 l.15 /possibly/, read /possible/

p. 11 l.9 /assumption, derived/, read /assumption - derived/

p. 13 l.1 /Mc/, read /McC./

p. 14 l.19 /the capacities/, read /for capacities/

p. 18 l.6 /and neither/, read /nor/

p. 18 l.11 /functions/, read /functions,/

p. 18 l.-5 /(...)/, read [...]

p. 25 l.16 /note 61/, read /note 62/

p. 26 l.-11 /apending/, read /a pending/

p. 27 l.-10 /tool in/, read /tool for/

p. 28 l.11 /importance being/, read /importance by being/

p. 30 l.-10 /on p.l./, read /on p. 14/

p. 31 l.-8 /in Greek/, read /into Greek/

p. 32 l.14 /first degree/, read /first degree problem/

p. 40 l.12 /abstract mathematics/, read /abstract mathematics and applied mathematics/

p. 41 l.15 /one or two branches/, read /one of two branches/

p. 46 l.18 /Method and Quadrature/, read /Method and Quadrature/

p. 46 l.-9 /circle of creative/, read /circle of creative mathematicians (or as no longer creative/

p. 47 l.-2 /mathematical philosophers/, read /philosophers' schools/

p. 49 l.3 /transformation of the mathematics/, read /transformation of mathematics/

p. 49 l.-2 /factor/, read /factors/

p. 50 l.15 /Academy'/, read /Academy,/

p. 51 l.2 /p. 36f/, read /p. 46/

p. 51 l.15 /ordering/, read /arranging orderly/

p. 53 l.13 /complicated/, read /complicated[195]/

p. 55 l.5 /around the lines/, read /around lines/

p. 95 l.-7 /astronomy", trigonometry,/, read /astronomy": trigonometry,/

p. 56 l.6 /sorts, one/, read /sorts. One/

p. 56 l.7 /Those which/, read /Those specimens which/

p. 59 l.-1 /were compared/, read /were, compared/

p. 61 l.5f /even if fixed and far from all books studied/, read /even if the body of middle books was not absolutely fixed, and far from all books were studied/

p. 61 l.-12 /curriculum; seemingly/, read /curriculum,- seemingly/

p. 61 l.-5f /evidence it was[226], rather some counter-evidence✓ read /evidence it was, rather some counter-evidence[226]./

p. 62 l.13f /the West (if it studied at all), one/, read /the West, one/

p. 64 l.1 /buildings./, read /buildings:/

p. 64 l.-1 /integrated in/, read /integrated into/

p. 65 l.-3 /the Muslims/, read /the knowledge of the Muslims/

p. 71 l.9 /culture (as the/, read /culture (viz. the/

p. 73 l.10 /graduation/, read /gradation/

Corrections to notes are located by means of note number, and lines counted from the beginning or the end of the note.

n.9 l.-7 /CAH I[i]/, read /CAH I[1]/

n.9 l.-2 /not argue/, read /not revolutionary. I shall therefore not argue/

n.19 l.-3 / "acre"). Than/, read /"acre") than/

n.24 l.-1 /Falkenstein (1937,46f)/, read /Falkenstein (1936,46f)/

n.25 l.-1 /(1930 B.C.)/, read /(c. 1930 B.C.)/

n.28 l.-2 /cises/cf. Powell 196, passim/, read /cises, cf. Powell 1976,passim/

n.34 l.3 /Kramer 1963, 62-69/, read /Kramer 1963, 62,69/

n.37 l.2 /Powell (1972, 1976, 418-422)/, read /Powell (1972; 1976, 418-422)

n.42 l.1 /Roover (1937, 1956)/, read /Roover (1937; 1956)/

n.59 l.8 /at a standard form/, read /to a standard form/

n.61 l.8 /uses to train/, read /used to train/

n.61 l.-3 /character of the/, read /character of the problem (two unknowns, eighth degree). Nothing is gained by the solution, neither greater methodical insight nor techniques applying to higher degree/

n.63 l.5 /n. 10.3/, read /n. 51/

n.63 l.-2 /and/, read /end/

n.69 l.3 /(a technical term)/, read /[a technical term]/

n.80 l.1 /1975/, read /1957/

n.82 l.2 /etwas Kassitisch/, read /etwa Kassitisch/

n.87 l.10 /"Mathematik"), however/, read /"Mathematik"); however/

n.96 l.-4 /type phenomenon/, read /type of phenomenon/

NB: From note 100 onwards the format for references changes to that used by the author in the manuscript, year and page number being separated by ":" instead of ",".

n.105 l.7f /(des punktes)/, read /[des Punktes]/

n.108 l.5 /Codes/, read /Codex/

n.112 l.-3 /done to the/, read /done to the phenomena of nature and craftsmanship in the/

n.126 l.1 /1966-75f/, read /1966:75f/

n.132 l.1f /(Geometrie ... Musik)/, read /[Geometrie ... Musik])

n.132 l.3f /(i.e. ... Grösse)/, read /[i.e. ... Grösse]/

n.133 l.-17 /mathematics harmonics/, read /mathematical harmonics/

n.136 l.2 /Commentary 65,/, read /Commentary 65, according to which Pythagoras made/

n.140 l.1 /Protagor/, read /Protagoras'/

n.144 l.-9 /(but ... length)/, read /[but ... length]/

n.151 l.4 /n.33.5/, read /n. 144/

n.179 l.1 /ibid/, read /Clarke 1971/

n.205 l.6 /1962:210-234/, read /1961:210-234

n.208 l.5 /the science/, read /the science of/

n.212 l.-2 /of neo-Pythagorean/, read /if neo-Pythagorean/

n.215 l.1f Should read /On the Arabic fractions, see Saidan (1974:368f), and the biographies of Abū'l-Wafā and al-Samaw'al mentioned in n. 214./

n.225 l.4 /Amir-Móez/, read /Amir-Móez 1959/

n.246 l.-4 /led/, read /lend/

n.250 /1964: 1954/, read /1964; 1954/

Corrections to items in the bibliography are located by author and year; when this method cannot be used, location is made by means of the preceding item.

Alster 1974 /Šuruppak/, read /Suruppak/

After Amir-Móez 1959: Insert /Anawati 1970. See addenda/

Arberry 1970 /(in Islam)/, read /[in Islam]/

After Archimedes: Insert /Aristophanes, The Clouds. See addenda/

After Baillet 1892: Insert /Banerji 1971. See addenda/

Benedict 1914/Cpmparative/, read /Comparative/

After Benedict 1914: Insert /Bergsträsser 1923. See addenda/

Boncompagni 1857a /I:II liber abaci/, read /I: Il Liber abaci/

Bose 1971 /Bose, D.S./, read /Bose, D.M./

CAH /1970./, read /1970- . Single volumes referred to e.g. as II[1], meaning Vol. II part 1./

Caratini 1957 /lunuless/, read /lunules/

Cassin 1966 /Jahrhunderts/, read /Jahrtausends. (Fischer Weltgeschichte 3). Frankfurt a. M:Fischer Taschenbuch Verlag/

After Chace 1927: The following item (Chace et al) was published in 1929.

After Childe 1971: Insert /Chuquet, Le Triparty. See Marre 1880a./

Corpus Juris Civilis /P. & I./, read /P. & I. Blaev/

Curtze 1897 /Algorism/, read /Algorismum/

After Denifle 1889: Insert /Diakonoff 1958. See addenda/

Drenckhahn 1951 /"A Contribution"/, read /"A Geometrical Contribution/

Eneström 1906 /252.262/, read /252-262/

Gardet 1970 /(in Islam)/, read /[in Islam]/

Grabmann 1941 /diviete/, read /divieti/

Hoernle 1883 /(...)/, read /[...]/

Jakobsen 1971 /Jakobsen/, read /Jakobson/

Jakobsen 1971 /33.37/, read /33-37/

Jayawardene 1976 /Jayawardene, S. S./, read /Jayawardene, S. A./

Jayawardene 1976 /the Renaissance/, read /the Italian Renaissance/

Karpinski 1910 /2+9/, read /209/

Lambert, M., 1956 /réformes/, read /'réformes'/

LdK /1975./, read /1975- ./

Marre 1880 /545-592/, read /555-592/

Marre 1880a /963-814/, read /693-814/

Needham 1954 /China I./, read /China. I- ./

Needham 1954 /Press, 1954./, read /Press, 1954- ./

Neugebauer 1935 /Quellen 3:3/, read /Quellen 3, erster Teil/

After Neugebauer 1935: Insert /Neugebauer 1935a, 1935b and 1937. See addenda/

Nissen 1974 /Späuruk/, read /Späturuk/

After Nissen 1974: Insert /Ooge 1926. See addenda/

Pines 1970 /(in Islam) ... 780-835/, read /[in Islam] ... 780-823/

After Pritchard 1950: Insert /Proclus, Commentary. See Morrow 1970 and Ver Eecke 1948/

Rodet 1881 /calculatuer égyptien"/, read /calculateur égyptien (Papyrus Rhind)"

Sayyili 1960 /Sayyili/, read /Sayili/

Sen 1971 /(in India)/, read /[in India]/

Sen 1971a /(in India)/, read /[in India]/

Staal 1979 /1979.80/, read /1979-80/

Sumerological Studies ... /1979/, read /1976

After Sumerological Studies: Insert /Suter 1889. See addenda/

Theon of Smyrna /Pupuis/, read /Dupuis/

Witzel 1932 /Kulturzentren/, read /Kultzentren/

INFLUENCES OF INSTITUTIONALIZED MATHEMATICS TEACHING ON THE DEVELOPMENT AND ORGANIZATION OF MATHEMATICAL THOUGHT IN THE PRE-MODERN PERIOD. INVESTIGATIONS IN AN ASPECT OF THE ANTHROPOLOGY OF MATHEMATICS.

BY JENS HØYRUP.

Introduction

For some years, the influence of the teaching organization at the École Polytechnique and in the post-reform Prussian universities on the changes in mathematical style occurring in the early 19th century has been an important focus of research. The close connection found in these institutions between part of their teaching and the advance of mathematical knowledge is rightly seen as a sharp contrast to the situation of the preceding period. It is then asked [1] whether the increasing systematization and rigour and the interest in the foundations of mathematics (the investigation of "Elementarmathematik vom höheren Standpunkt aus", to quote Felix Klein) was not at least in part a product of the new didactical connections of mathematical research considered as a social phenomenon. The answer seems to be that they are.

The following essay is an attempt to displace the question, in time and space, and to ask whether institutionalized teaching of mathematics has been of importance in other sociocultural situations as a factor influencing the development, style and cognitive organization of mathematical knowledge.

I have concentrated the investigation on mathematical traditions which have been important stages on the way towards modern mathematics: Mesopotamia, Egypt, Classical Antiquity, India, Islam, the Latin Middle Ages, and the early Renaissance. This choice was made for the most pragmatic of all possible reasons: My total lack of adequate knowledge outside this frame. I shall not try to defend my ethnocentrism by any other argument. Nor shall I make vain attempts to pretend that my knowledge of the area to which I have restricted myself is totally adequate. It is obvious to myself that it is not, and it will probably be obvious to the reader too.

In my title I claim that the investigation deals with an aspect of the anthropology of mathematics. This discipline does not exist. I choose the term because of dissatisfaction with the alternatives. History and social history of mathematics both tend as ideal types to concentrate on the historically particular, and to take one or the other view (or an eclectic combination) in the internal-external debate when questions of historical causality turn up. "Historical sociology" would point to the same neglect of the historically particular and a relativistic approach to the nature of mathematical knowledge which may be stimulating as a provocation but which I find simplistic and erroneous as it stands. [2]

What I looked for was a term which suggested neither crushing of the socially and historically particular nor the oblivion of the search for possibly more general structures: a term which neither implied that the history of mathematics was nothing but the gradual but unilinear discovery of ever-existing Platonic truths <u>nor</u> (which should perhaps be more emphasized in view of prevailing tendencies) a random walk between an infinity of possible systems of belief. A term, finally, which involved the importance of cross-cultural comparison.

The latter term suggested social anthropology, a discipline whose cognitive structure also seemed to fulfil the other requirements mentioned. That the term "anthropology of mathematics" would also imply the investigation of the relation man-society-culture-mathematics from the point of view of anthropology in general I shall only welcome.

I may not have succeeded in my attempt to live up to the requirements just defined. The essay may perhaps look like a piece of relativistic externalism. If it does, the reason is that the objective aspect of mathematical truth has been taken for granted in the exposition as a matter of course which needed no explicit mention.

If this is the case, as I suppose it partly is, it will be just one manifestation of a general characteristic of the essay. The approach is phenomenalistic and prospecting. What seemed to be the interesting aspects of relevant relationships is discussed briefly or at length. Other aspects are taken for granted in as far as they could be supposed known, even if they were strictly speaking part of the argument.

The phenomenalistic approach excluded the construction and discussion of a general theory for the relation between institutionalized teaching and mathematical thought. I would also suspect beforehand that teaching as well as mathematics are each on its own so profoundly embedded in other historical and social relationships that it would not be meaningful to expect that they could at all be isolated as a pair in a theoretically meaningful way.

Even if not directed by a general theory or hypothesis, my investigation was still guided by some overall assumptions. Some of these should be spelled out.

An important assumption is that even when number and measure have been introduced in human culture the search for mathematical coherence, the construction of explicit or implicit theories and of proofs, and even the use of mathematical argumentation are not inevitable. These are possibilities which have to be discovered and worked out in a complicated and not fully conscious process (a social process, as involving wider circles of persons, "mathematicians"). Furthermore, only if an adequate social soil is there will these possibilities be remembered and actualized. Inspired by the debate on the 19th century and by certain remarkable features of Sumerian mathematics I formed the hypothesis that institutionalized teaching might make up such a soil.

This, however, it must be emphasized, is only a guiding assumption. It will even turn out that institutionalized teaching (and mathematically fruitful teaching, for that) may hamper the development of demonstrations and argument instead of furthering them; and it will turn out that the coherence eventually produced by teaching need not be mathematical coherence.

The next assumption, derived from the phenomalistic approach - is that even though contemporary sociological concepts ("institutionalization", "profession") are only stringently adequate in modern society, they may when taken in a sufficiently loose sense be useful structuring devices in other social contexts - but that their adequacy must be investigated in each situation as a concrete question.

One of these concepts, that of "institutionalization", occurs in the title and in the central working hypothesis. The corresponding loose sense of the word "institution" might be expressed as a "relatively stable set of rules and established expectations."[3]

The final assumption to bring out is that the teaching affecting the development or organization of the mathematical knowledge of a culture will be teaching reaching into the adult age of the students. Knowledge which was once on the outermost front may in the long run be restructured conceptually in such a way that it can be taught to children. But in the same process it loses its directing influence on the total construction of mathematical knowledge. So, I have not tried to investigate teaching offered in childrens' schools.

The original version of this essay was a manuscript for a lecture given in the Bielefeld Institut für Didaktik der Mathematik in February 1980. Even if the text is changed in very many passages, I have tried to stick to the original form. Most closer investigations of my sweeping statements and all references to sources and secondary literature have been put into the notes. The result has been that many chapters contain more text in the notes than in the main text - and that the chapters which avoid this clumsiness do so at the cost of insufficient documentation.

A supplementary reason for the overweight annotation is the interdisciplinarity of the essay. Hoping to be read both by those interested in the general problem of the essay and by those who possess special competence in one of the cultures dealt with and who may therefore give me the necessary specialist's critique, I have chosen to combine supporting comment and documentation aiming at both groups.

May the reader be indulgent towards the resulting lack of style.

I shall close this preface by expressing my gratitude toward the mathematics teachers of Aalborg University Centre, who first provoked me to approach the question here dealt with; toward the co-workers of the Bielefeld IDM research project "On the relationship between the process of science and the process of education -- described by the example of the development of mathematics in the 19th century", who first invited me to lecture on the subject and afterwards urged me to write down the essay; toward lektors Paul John Frandsen and Aage Westenholz for, respectively Egyptological and Assyriological first-aid through several years on every occasion when I asked for it; toward Professor Wolfgang Helck of the Ägyptologisches Seminar in Hamburg for directing my attention to a number of sources; toward

Professor Robert Mc Adams of the University of Chicago for forcing me through critical questions to improve my discussion of the Old Babylonian period; toward the ever-patient staff of the Interlibrary Service of Roskilde University Centre; and toward my daughters who tolerated my concentration on the subject and two months' lack of care for them, for the laundry and for house-keeping in general.

The Sumerian beginnings

As so many other elements of our modern culture, mathematics came into being for the first time in Sumer, in Southern Mesopotamia. This happened in connection with the development of writing, around 3000 B.C.

By claiming that mathematics came into being in Sumer and in exactly this epoch I do not want to deny that Sumerian mathematics has its roots back in the Stone Age societies of the Near and Middle East, nor that these and other Stone Age societies were in possession of elements of mathematical thought. Many Stone Age peasant peoples have applied geometrical principles in construction techniques and for decorative purposes, and even geometrical play can be found.[4] In the Near and Middle East a system for arithmetical accounting related to the principles of the abacus (and to later Sumerian notation) was known as early as 8000 B.C. [5] Outlines of pre-Sumerian temple buildings were laid out in advance by strings and thus by use of geometry before the development of the earliest script [6]; metrological systems for lengths and probably even the capacities, used seemingly in connection with arithmetical calculations, were employed before the rise of Sumerian civilization [7]. What I wanted to express by my introductory phrase was, that only in the late fourth millenium B.C., when the first primitive writing was born, were all these different elements of mathematical thought moulded into one coherent system: MATHEMATICS.

It happened in Sumer [8]. The social context of the "event" (which of course was a process) was the incipient formation of the state, where a social elite concentrated around the temple used its key position in a number of important functions (the construction of irrigation systems, trade, exchange between the various groups of producers of the products of agriculture, herding, fowling, fishing and handicrafts, genuine ritual functions, and probably even more) as a base from where it could gradually secure for itself a politically ruling position, and simultaneously ensure for its own mouth the lion's share of that social surplus which was secured precisely through some of the social functions of the priestly elite [9]. The context was also that of the "urban" revolution, where the city rose to the position of a dynamic center from where development was determined, even though the city-dwellers remained in most cases a demographic minority [10].

The urban revolution and the rise of the incipient class-state do not constitute two mutually independent developments. On the contrary, they must be viewed as different aspects of the same social development, conditioned by the concrete natural, technological and social conditions prevailing in late fourth millenium Mesopotamia. In the concrete shape which they took on because of these conditions, they necessitated and furthered together the transformation of the above-mentioned token-based notation into a genuine, primitive script, including a numeral notation [11].

Until now I have not mentioned the school. However, the use of the proto-Sumerian script consisting of perhaps c. 1000 basic signs [12] was hardly learned by the future temple official just while following the footsteps of elder colleagues. From the earliest proto-Sumerian times there is ample evidence that organized teaching has taken place, presumably in the temple, and that the teaching methods current in later Mesopotamian schools were already in use [13]. The same evidence proves that the school was a place where knowledge was organized systematically, and that the schools of the single independent city-states were in mutual contact (since the development of the script and of numerical and metrological notations was the same even in far-separated cities [14]). So, the school was the organizer of knowledge in general and not just of writing abilities [15] (and probably even the organizer of a world view), in a way which was or at least tended to be both coherent and uniform.

Parallelly with the development of early writing and the systematization of knowledge expressed in word lists a coordination of the different elements of mathematical thought can be traced [16]. Metrological notations for entities which were hardly measured in that way in the token-system were developed (time can be mentioned [17]), and other metrological systems were extended according to arithmetical principles, among other things with fractional sub-units [18]. Area measures were constructed so as to permit the calculation of the area of a rectangular field from the product of length and breadth [19] (such a system may be less useful to the farmer and even to the taxator than a system based on natural units connected to sowing, ploughing or irrigation, but considered mathematically it is more systematic).

It is only a reconstruction, but on the other hand a reconstruction which makes sociological sense, that this organization of mathematics as a coherent whole (based on arithmetic as the uniting principle) is not solely due to practical "social needs" for computation; it is a fair guess that it was quite as much a natural product of that same school institution which in other domains acted as a systematizer of knowledge and cunning. Even if practical social needs not only for computation but even for systematization were present (which as far as the systematization is concerned has yet to be proved), it is more than doubtful whether practitioners acting without the background of an institution like the school would be able to elaborate it. So, according to my hypothesis, the creation of mathematics in Sumer was specifically a product of that school institution which was able to create knowledge, to create the tools whereby to

formulate and to transmit knowledge, and to systematize knowledge.

The proto-Sumerian society of around 3000 B.C. witnessed the humble beginnings of state formation, institutionalized schools and mathematics in the sense of a coherent body of knowledge and skills. During the following 1000 years all of these underwent important changes, and so did Sumerian society as a whole. Originally, the temple elites had used their position to secure for themselves economic wealth and political power, each in its own city-state. As a second step, a political elite headed by a king took over or was formed in the single city-states, absorbing even most of the functions which popular assemblies seem to have had in the early period, - and around 2400 B.C. larger empires began to be formed. Around 2100 B.C. this development culminated in the creation of a centralized despotic state ("Ur III"), where the state and its officials directed at least a very significant part of all economic activities [20].

As far back as official inscriptions tell us about the way in which the Sumerian state legitimated its existence, the power of state and king was justified by their asserted preservation of prosperity, justice and religious service [21], in close reflection of the social origin of the public authority. The only significant exception to this is an interlude from c. 2370 B.C. to c. 2230 B.C., during which Sumer was submitted to a non-Sumerian dynasty (the "Sargonides", from Sargon of Akkad its founder). The royal boastings of this dynasty concern military violence and the protection of foreign trade [22]. They are thus rather different from normal Sumerian ideology, but perhaps so much so that they remained without effect. In any case, the normal ideology of public authority had by Sargon's time been socially materialized since long by the rise around 2500 B.C. of a profession of scribes [23], conscious of its own importance, and reproduced socially and as far as professional consciousness is concerned in the school [24]. The importance of this profession could only increase with the advance of centralized empires, regardless of changing themes of royal boasting.

It should be noticed that the scribes were not identical with the priesthood, and neither with the body of higher officials. Priests as well as judges, provincial governors and kings were as a rule illiterate (until the breakdown of the Assyrian Empire only three Mesopotamian kings claimed to be literate [25]). None the less, the scribal profession was the carrier of very important social functions namely apart that of royal secretaries [26] the planning and management of most of the activities which had originally legitimized the ascendancy of the temple corporations (excluding, of course, the genuine cultic activities), and thus of activities which were central to the legitimating ideology of the state [27]. Presumably, the accumulation of these functions in the hands of a profession whose center of reproduction was the <u>school</u> must have enhanced the importance and prestige of the school. However, the teaching of mathematics belonged to the main tasks of the school [28], in agreement with the fact that mathematics was a tool necessary for a great number of scribal duties, especially those which ensured prosperity (be it prosperity of the population in general or prosperity of the elite - reality at least in Ur III seems to correspond to the latter possibility). Even ritually, the knowledge of mathematics was important: Gudea of Lagas (c. 2150 B.C.) writes about himself, that "Den Grundriss des Tempels entwarf er, gleich Nisaba (goddess of learning and <u>scribal art</u>), welche kennt die Bedeutung der Zahlen" [29].

There are various reasons (direct as well as indirect) for the assumption that the school and its teaching of mathematics came to play an increasing role as the body of scribes rose to

a profession [30]. Both have probably to do with the introduction of more advanced administrative routines in the same epoch and with a changing organization of labour [31]. The progress of centralized empires must be assumed to have added to the indispensability of a body of well-educated scribes, and the movement towards mathematical systematization already present around 3000 B.C. seems in fact to accelerate concurrently with the strengthening of the administration, the scribal profession and the school [32]. A decisive leap at least as far as mathematics is concerned seems to arrive during the Ur III period, when the requirements of the centralized economy for scribal training and precise accounting [33] grew tremendously - it is probably not a coincidence that Šulgi, the second and maybe greatest king of Ur III, claimed not only to be a god and to possess immense physical strength [34] but also that he was able to read, and verily that he was the "supreme scribe" of the goddess of writing and learning [35].

It is, indeed, exactly in early Ur III, as far as existing evidence can tell, that the gradual and slow systematization of the number system, of metrology and of accounting finished in a jump. From an early epoch, there had been a constant tendency for the ratio of 60:1 between successive units to gain foothold. On the other hand, different even if related notations were used to designate quantities of different sorts [36], and although the number system had since long become fully sexagesimal (in the same sense as that in which Roman numbers are decimal), it was no place value system. Rather early in Ur III, however, a sexagesimal place value number notation was created almost from scratch [37], differing from ours mainly by the lack of an indication of "absolut place" (corresponding to our decimal point), but maybe just for this reason immediately extended to fractions as well as integers [38]. Concurrently, the first accounting <u>systems</u> in the modern sense of that word, including balancing and automatic cross-checking, were created [39].

The sudden development of a "full" place value system is historically unique; only the Chinese have created a corresponding system independently of the Sumerians, and only in a process extending over many centuries [40]. The Maya, also working independently of the Sumerians, approached the place value system for integers, but probably they never grasped it fully [41]. But even the introduction of large-scale, systematic and uniform accounting practices is a remarkable achievement, compared for instance with the slow development and still slower general acceptance of double-entrance book-keeping in Mediaeval and Renaissance Western Europe [42]. Neither the sexagesimal place value system nor the book-keeping systems can possibly have been developed in a spontaneous process by practitioners in immediate response to the requirements of their scribal tasks, no matter how important these have been. They can only have come into existence as the products of some institution which was able to create, to coordinate, and to disseminate knowledge - and the only candidate for this is the scribal school [43].

In this connection it should be emphasized that the name of the institution which is here translated as a "school" was e-dub-ba, literally "house of tablets" (the corresponding word for the scribe was dub-sar). The delimitations of the concept of the tablet-house are far from clear, especially until the Ur III period. But it is probable that at least some tablet-house institutions had some sort of connection to royal chancelleries, and that they - like modern institutions of higher education - were concerned not only with teaching but even with creation and development [44].

From the beginning of Proto-Sumerian civilization until the end of Ur III, the influence of the school on Sumerian mathematics was (as far as we know it from published material [45]) restricted to the systematization of applied mathematics. Everything inves-

tigated up to now deals with problems which are not only (as it was sometimes to be the case in later Babylonian mathematics) disguised as practical; Sumerian mathematical texts are concerned with real "real-world problems". This does not imply that they are always realistic: One school text from the Sargonic epoch [46] deals with a field as long as 1297.444 km (given to that precision - the field might extend from the Gulf to central Anatolia); a problem of division from c. 2500 B.C. [47] distributes the contents of a silo as 164571 day-rations of a worker, with a remainder of 3/7 of a ration. In historical retrospect, this characteristic is typical of the teaching of even practically oriented mathematics when this teaching has been institutionalized and thereby has become the task of a partly closed milieu. It is related to the characteristics of pure mathematics in the sense that teaching of this type is concerned with the teaching of principles for calculations (maybe in the form of rote learning of unexplained recipes, but still principles). Only teaching traditions dominated directly by people who are not professional teachers of mathematics but primarily users of mathematics seem to be able to avoid this tendency.

The practical even if sometimes abstract character of Sumerian mathematics is in perfect harmony with what little we know about the curriculum of the Ur III school: It was purely utilitarian, and had no room for l'art pour l'art [48].

Babylonian culmination

The practical fixation of Sumerian mathematics and mathematics teaching may perhaps be regarded as a consequence of the integration of the Sumerian school and the scribal profession in the state administration [49]. We may guess that the unequivocal public attachment of the scribal function may have restricted the ideological autonomy of the scribal school and thereby its institutional independence.

This is only speculation, and we may leave it as it stands. In any case, the top-heavy bureaucratic Ur III empire broke down around 2000 B.C., probably by its own weight [50], and a new social structure developed which was more individualistic and in certain ways almost capitalistic [51]: The "Old Babylonian" culture, expressed first in clear form in the Larsa Kingdom and culminating in the "Old Babylonian Kingdom" under Hammurapi (18th century B.C.).

Individualism in the Old Babylonian society was not confined to the commercial sphere. It makes itself felt in the rise of personal correspondence dealing with private life, in religion, in the bulk of civil law documents, in the generalized use of personal seals, and even in literature and the plastic arts [52]. In general, it seems justified to speak of a new, more freely creative culture [53].

In this situation, the scribal profession seems to have become more independent as a social body; at least, it became less unequivocally attached to the public authority and function[54]. Possibly, even the school gained more autonomy [55]. At the same time [56], a genuine pure mathematics was developed, i.e. mathematics whose problems were not fetched from scribal practice but from <u>the challenges and possibilities created by existing mathematical knowledge</u>; mathematics, furthermore, based in part on <u>methods</u> with no relevance for down-to-earth practical tasks - derived, truly, from practitioners' methods, but transformed and developed by the contact with the theoretically generated problems.

One aspect of Old Babylonian pure mathematics is geometry, almost

solely in the form of calculating geometry. Anyhow, most Old Babylonian geometry was practical, and the small remainder (derived, seemingly, from interest in geometrical decoration) never went very far [57], nor did it imply as far as we can see any development of advanced methods [58]. Clearly, the most important aspect of Old Babylonian pure mathematics (and the aspect which distinguishes all Old Babylonian mathematics from its Sumerian background) is the dominance of <u>algebra</u>, i.e. verbally expressed "equations" with one or more unknowns, most often of the second degree (but sometimes of the third and in cases even of the fourth, sixth or eighth degree) [59]. Even exponential problems occur, but whether the two texts in question should be classified as pure mathematics seems doubtful - to me they look like rather practical computations of interest [60].

Not only the problems but even the methods may legitimately be called algebraic, since the solutions are worked out step by step by procedures which we would characterize as algebraic: substitution of variables, reduction, etc. Of course, no symbols are used; as far as can be seen from the texts, we have mostly to do with highly standardized "rhetoric algebra", of the sort current until the late middle ages.

The problem of the character of the two exponential texts touches an important aspect of Babylonian pure mathematics: It <u>looks applied</u>. One has to penetrate behind the entities which appear (length and breadth of fields, quantities of earth, dimensions of triangular fields to be divided up between brothers, etc.) and into the mathematical character of the problems; if one does this, many problems turn out not to be practical at all (not even in principle, as in the Sumerian school exercises mentioned above); they can only be theoretical exercises, puzzles whose main merit is that they <u>can be solved</u> [61].

Hypothetically, I would explain this very characteristic form [62] of Old Babylonian mathematics by its double connection to the school and to the scribal profession. The argument goes like this: In spite of all cultural shifts and all individualism, even the Old Babylonian states were legitimized by the traditional Sumerian "welfare-state-ideology" of justice and prosperity. This is evident for instance from the prologue and epilogue of Hammurapi's famous law-code [63]. From many sources we also know about the professional pride of the scribes [64]. There are good reasons to believe that this pride must have been at least in part connected with those scribal functions which were central to the state or to the maintenance of civilized society and which were therefore highly esteemed - and apart reasons, there is quite a lot of evidence [65].

Now, in the fulfilment of these functions practical, applied mathematics played an essential role [66] - and for this reason it would be strange if competence in the handling of difficult mathematics were not an important element in the scribe's professional pride - the more difficult the better, since the understanding of complicated mathematics would obviously distinguish the competent scribe from everybody else in society. And indeed, several texts show that mathematical skill was a reason for scribal pride [67], and one text listing the culmination points of scribal cunning [68] closes on mathematics [69] and the use of musical instruments - two fields which according to the text are considered even more secret and inaccessible than secret writing, occult and technical languages of the different crafts, and notarial competence.

If we leave mathematics for a moment and look at other parts of scribal cunning we will find what looks like an important change from the Ur III to the Old Babylonian edubba. The Ur III scribe was proud of being able to perform his practical tasks with skill - this is at least what seems to follow from the

Šulgi hymns and from the early "School-days"-composition [70]. In contrast, the Old Babylonian scribe was taught much which had little practical value [71], and he was taught that these matters were among the most important constituents of the specific scribal quality called "humanity" [72]. Practical scribal skill was no longer sufficient, virtuosity extending far beyond the useful even if extrapolated from basic skills had become the most important foundation of scribal self-consciousness.

The Babylonian pure mathematics seems to follow exactly the same pattern. Indeed, it seems to be the product of a desire to display high skill by treating successfully as difficult problems as possible. It tended to investigate problems chosen neither for their practical interest (as was the case in earlier Mesopotamian mathematics) nor for their inherent or theoretical interest (as in Greek and later pure mathematics, cf. note 61) but just according to the possibilities they offered for displaying skills belonging to an exclusive scribal corporation. However, due to the fact that this took place inside a school institution where continuous tradition (and a marked care for tradition) permitted a gradual spiraling accumulation of new results and new methods, the resulting product was not a heap of isolated virtuoso's tricks but instead an impressingly coherent science of algebra [73].

Why, then, does Old Babylonian mathematics look so applied? Why disguise a problem of proportionality resulting in a second-degree equation as a question concerning an anti-fortification ramp [74]? The answer could be that high mathematical ability was only relevant to professional pride because of its ultimate connection to professional activities and the esteem adhering to the social function of the scribe. What professional pride needed was not pure mathematics in the Greek or later sense, it was "pure applied mathematics". Put in another way, our post-Greek concept of mathematics is anachronistic when applied to Babylonia. Mathematics which is formally pure, i.e. which deals with abstract mathematical entities, was only created by the Greeks. The Babylonians did not possess it, and they had no incentive to invent it - what their professional pride needed was something more complicated than the calculations of daily applied mathematics, but on the other hand something still belonging to the same species. The Babylonian might develop mathematics which was pure as far as substance is concerned; but the form had to remain applied - to formulate the matter in an Aristotelian framework.

In part, we should notice, the applied look of Babylonian mathematics is also a question of translation. In fact, Babylonian mathematics made use of a highly stereo-typed technical terminology going back at least for a part to Šuruppak [75]. Of course, literal reading of a vocabulary of Sumerian origin will point to the practical concerns which defined Sumerian mathematics. However, by Hammurapi's time the terminology had been fixed for many centuries, and it is therefore quite possible (and probable) that it had lost most of its practical connotations to the Old Babylonian users. After all, we rarely think of apending plumb line when speaking of a "perpendicular". In the same way, one should work only a few days with elementary Babylonian algebra problems before he thinks automatically of "length", "breadth", "field" and "earth" as "x", "y", "z", "area", (i.e. "x·y") and "volume" (i.e. "x·y·z").

Such a fixation and stereotypization of language must itself be seen as a product of the school institution. Since the stereotypization makes the distinction between formally pure and formally applied mathematics meaningless in the range of algebraic standard problems (the domain which was totally dominated by stereotyped expressions), we may say that the

didactic and tradition-making impetus of the school institution was in this domain able to break through the barrier towards formal purification raised by the demands of professional self-importance. Only in the more complicated problem-constructions does this barrier really make itself felt; coincidently, it is just in these constructions that the substantial purity may often get its most striking expression.

Until now, only the pure tendencies in Old Babylonian mathematics were discussed. This is due to the special viewpoint on which I concentrate: The influence of mathematics teaching on the development and organization of mathematical knowledge. If we disregard this special interest we will have to change the emphasis. The pure tendencies remain important (especially if we include the elementary second-degree problems as I think we ought to); but they are no longer all-important, not even dominating. The larger part of Old Babylonian mathematics was just as practically orientated as its Sumerian ancestor had been. Nothing could of course be less strange. Primarily, the scribe was a practical man; professional pride could only be a secondary phenomenon dependent on social importance and esteem due to the practical function. Primarily, the edubba served the formation of practical men; only because it did so could it uphold a certain degree of intellectual autonomy. And primarily, mathematics was a tool in the scribe's practical functions; only because it was an important and efficient tool could it fulfil a secondary task in professional ideology.

Obviously, a social interpretation of Old Babylonian mathematics like the above can hardly be supported by anything but indirect arguments. However, to the positive arguments already given one can add others, more negative in their character, dealing with what happened after the dissolution of the Old Babylonian kingdom around 1600 B.C. A warrior people (the Kassites) conquered the country and transformed the state into an effective and undisguised machine for war and exploitation [76], with no ideological legitimation whatsoever founded on justice and general prosperity. The literary production of the time (which was as before a product of scribal culture) demonstrates clearly to what extent this was a cultural shock to the indigenous scribal elite which was now as before in charge of public management. Contemporary literature expresses pessimistic "internal emigration", admiration without limits for the glorious past and for tradition, and pious religious coming to terms with the sufferings of the just [77]. The scribal profession hardly lost in importance being harnessed to the Kassite military machine - on the contrary, the Kassite state once more became a centralized "palace economy" [78]. But this new role (which was new at least as far as ideology is concerned, less so perhaps in reality) seems to have offered no reason for professional pride. Scribal self-consciousness was put on another foundation, namely that of belonging to an ancient and most glorious tradition [79]. The scribal school disappeared as an independent institution; instead the apprentice scribe was adopted (or born?) into a "scribal famliy" [80]; by this change, the body of scribes was transformed from a corporate profession, proud of its social function and ability, into a sort of hereditary aristocracy predominantly conscious of descent and tradition. To make the change complete, the hitherto strictly secular scribal occupation now became intermingled with priestly functions; on the other hand, the elaborate division of the scribes into 15 groups of occupational specialization vanished, to make place for indefinite scribal "sages" [81].

What makes these details of social and cultural history interesting in connection with the anthropology of mathematics is that they were accompanied by a loss of interest in mathematics. The Kassite period presents us with no or almost no problem texts [82], only with those table texts nec-

essary for the accomplishment of practical calculations [83]. Apparently, the practical tasks incumbent now as before on the scribes could easily be carried out without the battery of Old Babylonian algebra (indeed, these tasks were the same in the Kassite and the Old Babylonian states) [84]. Practical tasks alone could not keep alive the interest in high-level mathematics. So, if we look for social factors behind the development of Old Babylonian mathematics we will have to look for social factors which disappeared in the Kassite state - and here the combination of the edubba-institution with a professional self-consciousness based on agreement between the scribal function and the ideological legitimation of the state comes to mind.

Strictly speaking, the disappearance of mathematical interests in the Kassite period cannot have been complete. Though rarefied beyond detection, the tradition must have been continued to some extent, since Babylonian mathematics knew a revival in the late first millenium B.C., and since Diophantine as well as Islamic algebra is obviously connected with the Old Babylonian algebra. However, this changes nothing fundamentally in the above argument. For this reason, and because the late Babylonian mathematical revival seems to offer nothing of interest as to the influence of didactics on mathematical thought [85], I shall leave the treatment of Mesopotamian mathematics at this point.

What about Egypt?

One reason to treat the development of Mesopotamian mathematics as extensively as I have done is of course that there seems to be so clear connections between the school and the rise of mathematical thought. Another reason is the fundamental importance of Mesopotamian mathematics for Greek, Indian and Muslim and so also for modern mathematical thought.

The other pre-Greek mathematical tradition influencing us in that way is that of Ancient Egypt [86]. Will an investigation of the development of mathematics in Egypt provide us with a parallel concerning the influence of the school?

I would answer by a hesitating "yes". Hesitating because the sources are by far too few to allow the writing of a well-substantiated social history (or just a real history) of Egyptian mathematics.

Egyptian mathematics developed independently of Sumerian mathematics [87], at most a few centuries later. At the very beginning of the first dynasty, c. 3100 B.C., the system of integers is fully developed, and a canonical system for pictorial representation of the human body, making use of a network of squares related in spirit if not by direct tradition to those used by Leonardo da Vinci and Albrecht Dürer, turns up fully developed [88]. Presumably, a number of metrological systems were in existence at this time, without being yet put in mutual arithmetical connection [89]. So, in the sense defined on p.1. elements of mathematical thought existed but MATHEMATICS had not yet emerged.

At the apex of the "Old Kingdom", around the mid-third millenium B.C., it probably had. The originally independent metrological systems had probably been brought in arithmetical connection with each other, and a number of applied mathematical problems had seemingly been mastered: complicated distribution of rations, area measurement, and various calculations concerning pyramids [90]. This took place in a context where a centralized royal power had been established,

supported by a staff of scribes in possession of the rather newly developed hieroglyphic script as well as of considerable social prestige [91]. Further details, especially about the education of the Old Kingdom scribes [92] and their use of mathematics, is scarcely known. We can only say that any field inside Egyptian mathematics whose presence already in the Old Kingdom context can be reasonably supported by direct or indirect evidence seems to belong to the domain of professional scribal practice.

Certain Egyptologists tend to agree with the Ancient Egyptian tradition and ascribe an Old Kingdom origin and full development to almost everything in later Egyptian culture; so, even mathematics is estimated to have reached its full development, as found in sources from the "Middle Kingdom", (c. 2000 B.C.) and later, as early as c. 2500 B.C. [93].

It is my impression [94] that this estimate is not quite correct. Truly, metrological systems, fundamental calculational methods and the mathematization of current practical problems may very well go back to the Old Kingdom. But the theoretical unification of Egyptian mathematics as a fully coherent body of thought (as found in the great mathematical texts and especially in the Rhind Mathematical Papyrus) I would only locate in the Middle Kingdom. From this time on, and until the disappearance of the tradition in the 7th century (A.D.) [95] very little happened to Egyptian mathematical thought, apart some changes in terminology (and a final translation in Greek, which had become the administrative language). The few conceptual changes which did take place during these almost three millenia I would describe as a "creative dissolution", where a few innovations (most important the introduction of factorization methods and multiplicative concepts) were grafted on the original system (which was a strict structure built on additivity, aliquot parts and what I for lack of better words would call scaling, i.e. the measurement of one number by another one), without ever being so fully integrated that the system came to combine the increased flexibility with a theoretical coherence comparable to that of the Middle Kingdom texts [96].

The very distinctive and very stable character of Egyptian mathematics as a strict system based on elementary concepts like additivity and aliquot parts has earned it the fame of being primitive. Indeed, it is one of the recognized commonplaces of the history of mathematics that while Old Babylonian mathematics invented most of what became known in algebra until A.D. 1400, and the scribes of Ur III knew at least the full place value system, then the Egyptians got stuck in a complicated system of unit fractions. A simple first degree like $\frac{97}{42} \cdot x = 37$ would in quasi-Egyptian notation rather look [97] $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 37$ (and could be expressed no simpler). The solution (which requires quite much space, and a fair amount of ingenuity and training) can be translated $x = 16 + \frac{1}{4} + \frac{1}{28} + \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$. True enough, the system seems cumbersome and primitive compared to the Mesopotamian place value system, where equivalents of this problem would be left out as trivial intermediate calculations.

Can this be brought in harmony with what we know about the social context of Egyptian mathematics and with the above hypotheses about the social roots of Old Babylonian mathematics? After all, in spite of great differences between Egyptian and Mesopotamian society, both areas were provided with bodies of scribes the practical tasks of whom were rather similar (and indeed, the practical problems dealt with in Mesopotamian and Egyptian mathematics are very near the same). The social positions of the scribes in the two societies were similar too; they constituted, as one Egyptologist has

put the matter, "a 'white-kilt' class, people who would not need to soil their hands or garments with work" [98]. Even Egyptian scribes were taught in school about their particular social standing [99], and they were most proud of their art [100]. Egyptian scribes used mathematics in their job just as much as Mesopotamian scribes, and here as there mathematical ability was a key ingredient in professional pride [101].

A strong social determinist might then wonder why Egyptian and Mesopotamian mathematics are as different as in fact they are. He would be right, for nothing could emphasize the limits of social determinism in the history of mathematics better than a comparative analysis of Egypt and Mesopotamia. However, my hypothetical explanation of the character of Mesopotamian mathematics was not meant as one of blind and total social determinism. Let us therefore try a comparison point by point in order to investigate the reach of explanations by social function, professional basis, didactical organization and professional ideology.

1. The Egyptain unit fraction system is more cumbersome than calculation by the Mesopotamian place value system. But when you have got used to it and when you have adopted the ways of thought behind it you will find that it is no longer that cumbersome. The Egyptian mathematical structure could easily fulfil its job in practical management (we know it did). So, we have got another demonstration (apart the Kassite and second millenium Syrian cases) that the effectiveness and range of Old Babylonian mathematics went far beyond what was necessary for practical purposes, and that these alone can not have given rise to the development of Old Babylonian mathematics.

2. On the other hand, there is one fundamental common char-

acteristic of Egyptian and Mesopotamian mathematics: The status of arithmetical calculation as the uniting principle. Obviously this agrees well with the principle purpose of mathematics in both societies: That of being a tool for scribes in the fulfilment of their practical tasks.

3. The restructuration of pre-existent elements of mathematical thought into a system of MATHEMATICS seems even in Egypt to have taken place in a situation where a class of professional managers had started running a centralized state. Surely, in Egypt this state was much larger than the proto-Sumerian and early Sumerian city-states. Concomitantly, it seems plausible that the level of Old Kingdom mathematics as revealed e.g. in monumental buildings was well above that of contemporary Sumerian mathematics.

4. The character of the unit fraction system (enigmatic to the common man even if he happened to understand the notation, difficult to learn also for the scribal school pupil, but easy to handle for the initiate) made it just as good a support for professional pride as the algebra of the Old Babylonians. It is also clear that even though the unit fraction notation was used for administrative purposes from the Middle Kingdom onwards its precision was far beyond the necessary: who would really be able to control if the temple worker received the last $\frac{1}{180}$ of his jug of beer - the scribe at least did not care, he did not even add up his (wrong) numbers in order to check them [102]. For practical management, metrological sub-units were much easier to handle than unit fractions (just as, by us, decimal fractions are much more useful for practical purposes than common fractions with an arbitrary denominator). The elaborate unit fraction system came from mathematics, not from practice.

5. The refinement of Egyptian mathematics, its structuration into a coherent whole, obtained not least through the development of the elaborate unit fraction system, was (according to the above arguments) contemporary with a bureaucratic centralization of Egyptian society (the Middle Kingdom), following upon a period of decentralization. Above all, it was (according to this same chronology) contemporary with the establishment of a system of government schools [103] (exactly those where the didactical texts inoculating professional self-consciousness were used). Even though we know next to nothing about the mathematical teaching of these schools it is at least possible and in my opinion plausible that it was organized teaching in these schools which created the need and impetus for the systematic restrucuturation of existing mathematical practice.

6. On the other hand, the positions of scribes were not identical in the Old Babylonian Kingdom and the Egyptian Middle Kingdom. At least until the first millenium B.C. the Egyptian body of scribes remained less independent as a profession and more unambiguously bound to the public service than the case had been in the Old Babylonian Kingdom. This circumstance may have contributed to bind Egyptian mathematics more closely to immediate utility than Old Babylonian mathematics had been - cf. above on the practical fixation of Sumerian mathematics. In any case there is an important difference between the higher levels of Egyptian and Old Babylonian mathematics: Advanced Old Babylonian mathematics (higher-degree algebra) looks applied but was not; it was a jump into new mathematical realms and had no place in daily practice; advanced Egyptian mathematics (as far as it was expressed in specific techniques like the full unit fraction system and was not only an abstract coherence) was applied in practical management. In other words: Advanced Old Babylonian mathematics was a free activity, relatively unbound by what went on in practical applications;

Egyptian advanced mathematics consisted in an advanced treatment of practical applied mathematics.

Before we evaluate the outcome of the comparative analysis we should remember that belief in total social determinism is absurd even for reasons of principle. Social factors (be it application and needs, schools systems, professional organization or professional ideology) act on a substratum of preexistent mathematical conceptions, techniques and notations. The final development depends on this substratum just as well as on the social factors involved (which may be more complex than subsequent historical investigations will reveal) and on random accidents (e.g. the intervening persons). When mathematical specificities are dealt with the substratum and the accidents are all-important - social factors cannot determine whether a decadic or a sexagesimal number system is created. In questions involving the global character and organization of mathematical thought social factors may be more dominating.

So, I will not wonder that two cultures with independent and rather different mathematical background and far from fully identical social contexts gave rise to different mathematical traditions. I should rather propose as the conclusion to draw from the comparative analysis, that the development of Egyptian mathematics constitutes as close a parallel to that of Mesopotamian mathematics as can reasonably be expected - not least concerning the influence of the interplay scribal profession/scribal school. Least close is the parallel (as is to be expected) in the domain traditionally most investigated by historians of mathematics: that of specific mathematical methods and techniques.

Classical Antiquity: Abstract mathematics and liberal education.

During the last century it has become increasingly clear that Mesopotamian and Egyptian mathematics are important roots for our own mathematical thought. Greek [104] mathematics, on the other hand, is no root. It is part and foundation.

So, what distinguishes Greek (and later) mathematics from its Bronze Age forerunners?

Everything, the positivist would answer, in so far that he would at all admit the existence of Greek mathematics as a meaningful entity. Everything, from the age and the language to the tiniest detail.

Disregarding nominalism and positivism, we may ask once more: Which essential characteristics distinguish Greek mathematics from the Vorgriechische Mathematik?

Essences are always subject to discussion. I for my part should point to four features of Greek mathematics of which three are commonly recognized as characteristic.

First, Greek mathematics is formally pure, i.e. it concerns itself with abstract, idealized entities. And it knew it did [105].

Second, it is rational, and at least from the fourth century B.C. onwards it is deductive and axiomatic. In general, it expresses itself in general rules and not through specific examples [106].

Third, geometry is not just another field of application for arithmetical calculation (as was more or less the case in the bronze age cultures). Geometry stands on its own, at least on a par with arithmetic.

Fourth, Greek mathematics is pure also in substance, i.e. concerned with mathematically defined problems irrespective of their practical relevance. Its problems were, however, not chosen for the merit that they could be solved by virtuosity and methods at hand, as was the tendency in Old Babylonian mathematics. They were rather chosen for some sort of inherent interest - we may perhaps say that they were not chosen to satisfy a wish to show one's abilities but rather from curiosity and a search for theoretical perfection [107].

Much of this can be connected to the social organization and the general cultural and philosophical climate of the Greco-Roman world. That matter we shall not investigate in general. But we shall come across a number of its facets while trying to sort out this problem: How far was the specific character of Greek mathematics a consequence of the organization of education (or educations)?

In the preceding chapters it was argued that both great Near Eastern bronze civilizations are instances of important influence from the practical use of mathematics on its development as a science, even so far that the schooling and social role of the practitioners gave rise to theoretical developments beyond the practical needs. In the world of Classical Antiquity, no such influence can be traced.

Of course, mathematical practitioners still existed, even if the prestigious scribal profession had disappeared [108]. Most applications of mathematics were still necessary, and even on an increased scale - detailed public finance like that of Ur III is the most important exception [109]. As we have already seen

in several connections, however, practical management in the Ancient world required only a low level of mathematical knowledge. On this level, certainly "all ... inventions (directed to the necessities of life) were already established", as Aristotle put the matter for technology in general [110].

The only influences from mathematical practice on the development of mathematical thought seem to be of other kinds. One is the theoretical speculation inspired by the ways of the practitioners (be they Greek or foreign [111]). This may be the origin of some of the geometrical theorems ascribed to Thales of Miletus [112]. Similarly, the figurate numbers of the Pythagoreans could be derived from the arrangement of the pebbles of an abacus [113], and their interest in "perfect numbers", i.e. numbers which are identical with the sum of their aliquot parts, may be derived from observations of the Egyptian methods of arithmetical calculation [114], which were surely known at least to Plato [115]. Even the Archimedean Sand-Reckoner may be a sort of extrapolation from the uses of common calculation [116]; in any case his way to calculate and express the circumference of the circle demonstrates familiarity with the discipline of logistics (practical calculation) [117]. Diophant's algebraic Arithmetic is probably dependent on Mesopotamian algebra [118] via connections of which nothing is known but which may of course have to do with the descendants of scribal training in the Near East. Finally and most important, some of Hero's works demonstrate a clear interest in the working of practitioners; however, the introductory chapters of the Metrikon [119] as well as the Dioptra [120] indicate that Hero's main aim was to develop a discipline of applied mathematics (and applied science in general) from high-level mathematics, in order to improve on the bad methods of working practitioners. All in all, there is little trace of influence from practical mathematics on the development of mathematics as a science, and no trace at all of an influence from the training of practitioners.

This does not necessarily imply that institutionalized teaching in general did not affect the development of Greek mathematics. In fact, the Hellenic and Hellenistic world created a new kind of education, paideia or "liberal education", the aim of which was to form the citizen and the person, not to provide him with skills to use in practical life [121]. Since there are obvious parallels between this educational ideal aiming at completeness and the specific character of Greek pure mathematics; since furthermore almost every Greek text dealing with the distinction between abstract mathematics emphasizes the higher value of theoretical geometry and arithmetic as compared to geodesics and logistics used by practical people [122]; since, finally, mathematics played an important role both in the mature liberal curriculum, in one of its educational predecessors (cf. below) and in Plato's influential [123] advocacy of paideia; the hypothesis suggests itself that the spirit of Greek mathematics was formed by Greek paideia. But the hypothesis should be thoroughly checked since both might equally well be product of the same social and cultural forces [124].

Two related but different kinds of education lead forward to the liberal education of late Hellenic and Hellenistic times. One is the Pythagorean school of which later sources affirm that it was secret. The other is the "free" teaching of philosophers culminating with the sophists ("free" in the sense of a low degree of institutionalization, far from always gratis).

Of neither is very much known, especially not when the connections to mathematics are concerned. As to Pythagoras and

the Pythagoreans so much seems sure that Pythagoras founded a brotherhood in Croton in Southern Italy around 530 B.C., and that this brotherhood spread rapidly over the Greek cities of Southern Italy [125]. It is possible but by no means assured that the brotherhood was secret at least with respect to its teachings [126]. However that may be, the Pythagorean brotherhood gave rise to a continuous community which still existed at Plato's time [127], and inside which a number of religious and philosophical doctrines were handed down through some sort of more or less organized teaching. Probably from the beginning [128], and at least as traditions had been formed at Plato's time [129], mathematics was the backbone of the Pythagoreans' philosophical teachings [130]. We are sure that the initiates of the school (or perhaps either an inner circle of initiates or one or two branches of the movement) [131] were called mathēmatikoí, derived from a verb meaning "to learn", and it is probable that it was the Pythagorean school which gave rise to a shift of meaning of the word mathema, from "learning" and "knowledge" in general to that of "knowledge of number and magnitude" [132], i.e. mathematics. Finally we know that a good deal of mathematics [133] and numerological speculation was developed by the mathēmatikoí and kept by them as a tradition [134] (maybe secret, maybe not).

So, we know that institutionalized teaching took place inside the Pythagorean brotherhood; we are as sure as one can hope to be that mathematics got its name in the Pythagorean circle as the "matters being learned". We know that mathematical "research" was pursued by some of the Pythagoreans, and that a mathematical tradition was built up from the mathematical problems investigated and the methods created. It is a reasoned guess but still a guess that the transmission of this tradition took place inside or partly inside the framework of the existent institutionalized teaching [135], and that a main incentive for the further development and organization of mathematical knowledge was this connection to the teaching system of the order. [136] Indeed, had it not been for the requirements and possibilities of reasoning mathematics teaching [137], might not the mathematical and numerological elements of the original doctrine have remained untouched and undeveloped [138]? As already mentioned, this is conjecture; in view of the state of the sources it is probably bound to remain nothing but conjecture.

The other root of the mature liberal education is the "free" teaching of the philosophers. Obviously it was far less impregnated with mathematics than the teaching of the Pythagoreans. Still, we should not dismiss it from our field of interest without a somewhat closer inspection.

As far as organized teaching is concerned, the sophists must be regarded as the culmination of this tradition [139]. Their aim was not the knowledge of esoteric truths for the chosen few, as that of Pythagorean teaching can be formulated. Their purpose was man, and more precisely political man, man living in a city-state. So, they did not have the Pythagoreans motive for far-ranging interest in mathematics. Still, the aim to produce "better men", "better family-heads" and "better statesmen" [140] did not exclude interest in many fields which seem to us to have no direct relevance for such an educational purpose, including both philosophy of nature [141] and mathematics [142]. However, the sources bearing witness of sophist mathematical activity seem to point to an influence from the practical [143] orientation of the sophist teaching going in the opposite direction of the general tendency of Greek mathematics: Towards a phenomenalist and maybe discursive approach, away

from logical purism, abstraction and axiomatics [144]. At least with the sophists, the analogy between the search for the complete and coherent citizen and for completeness and coherence in mathematics turns out to be nothing but analogy.

Other participants in the non-Pythagorean philosophical movement were more interested in mathematics, not only the halfway legendary Thales but even philosophers whose mathematical works are subject to less doubt - e.g. Democritus [145]. Surely, these philosophers taught [146]. But there is little in the history of early non-Pythagorean Greek philosophy which looks like being organized around a teaching tradition. Nothing indicates that teacher-student-relations were anything but personal ties, nor that the books indubitably written went into a stable teaching tradition. So, these philosophers taught and wrote what they thought and discovered. They hardly thought or discovered in a way directed by, organized by or aiming at institutionalized teaching.

On the whole, the same seems to hold good for the little we know about the mathematical investigations and writings of the non-Pythagorean philosophical tradition. Only one possible exception can be mentioned - but that a most important one, to be sure: what amounts perhaps to a tradition for writing Elements, i.e. surveys or textbooks covering the fundamentals of mathematical knowledge ("those geometrical propositions, the proofs of which are implied in the proofs of the others, either of all or most" [147]), and organized presumably in a systematic and more or less deductive manner.

The first collection to be recorded is that of Hippocrates of Chios [148] (fl. c. 440 B.C. [149]). We may guess - but since all such early Elements are lost we can only guess - that this tradition (if it was one) arose as a response to the need for some text-book in an at least weakly organized teaching system of an environment of mathematically oriented philosophy [150], and that it may have been an important factor in the transformation of what was already abstract and rationally arguing mathematics [151] into a deductive and axiomatic system [152] as known to us not only from the Euclidean Elements but also from Aristotle's discussions [153]. At least, if abstract mathematics built on arguments is to be represented in textbook form, then the inherent impetus of the process will be one leading towards axiomatization in the Euclidean sense: The systematic organization of fundamentals in one text will make conspicuous all circular or incomplete arguments, and the attempt to get rid of such flaws will lead towards something like the isolation of hóroi ("delimitations", i.e. "definitions"), aitémata ("requirements", i.e. "postulates") and koinài énnoiai ("shared conceptions" i.e. "axioms") [154].

It must be emphasized that the evidence supporting this hypothesis is mostly of a rather indirect character, at least until the time of Aristotle. As in the case of the early Pythagorean development of mathematics it is nothing but a plausible possibility that didactical concerns and didactical practice were among the active forming factors of Greek deductive and axiomatic mathematics - and the state of the sources is such that one can presumably not procede very much further.

One should also emphasize that Pythagorean teaching as well as the hypothetical schools of philosophical mathematics forming background to the Elements-tradition were institutions intended as far as one can see for mature men. Neither was a genuine

paideia in the sense of education of the youth.

After leaving its various precursors, we should now turn to the "mature" liberal education of later centuries (i.e. from Plato's epoch onwards). Since my main conclusion will be that it offered nothing decisive to the development of mathematics I shall permit myself to discuss it without historical subdivisions.

The level of the primary school was - as far as mathematical ambitions are concerned - quite low. The children were taught the notation for integers, how to use the elementary pseudo-fractions (a system of fixed sub-units, each having its own name: 1/12, 1/6, 1/4, 1/3, etc.) and perhaps the most basic part of the addition and multiplication tables [155].

What might be characterized as the "secondary education", given to adolescents [156], was from Plato's time onwards more or less neatly built on the 7 "liberal arts": Trivium, the literary arts grammar, rhetorics and dialectics, and quadrivium, i.e. the four Pythagorean mathēmata arithmetic, geometry, astronomy and harmonics. So, mathematics was (at least nominally) an important constituent part of secondary education. However, the aim of the liberal education was to impart general culture - "for gentlemen", so to speak. In view of this and of the restricted mathematical background given in primary education it is no wonder that the secondary education level of the mathematical arts was far below the advanced developments of contemporary mathematical science. No better proof of this can be found than the level of introductory mathematics teaching given in the philosophical schools to students who had already passed secondary education, and who had apparently passed it with better than average results since they had opted for further study. The manuals and compendia used on this level are at their best honest popularizations [157], but often verbious and semi-mystical descriptions without proof of the basic facts of Pythagorean arithmetic.

Accordingly, there is no reason to assume that the liberal teaching system from 400 B.C. onwards had any direct influence on contemporary progress in mathematical knowledge - whether we think of deductive, pure geometry, of advanced Euclidean arithmetic (beyond what had been created by the Pythagoreans until the time of Plato and Archytas), of Hero's writings on applied mathematics, or of Diophantine algebra. The advances were as far as can be seen from the sources due to a circle of free full-time amateurs and (in Hellenistic Egypt) scholars paid by the state - people who felt engaged in sort of professional community. The existence of such a circle seems at least to be implied by the letters which introduce Archimedes' Method and Quadrature of the Parabola [159] and book 1, 2 and 4 of Apollonius' Conics [160]. Corresponding introductory letters by Diophant [161], Pappus [162] (with a certain reserve) and the 6th-century commentator Eutocius [163] indicate that there existed on the margin of this circle of creative descendants of this circle) a group of non-creative amateurs who were tied to mathematics by personal relationships to greater mathematicians, not via any organized teaching. Letters by Archimedes and Eratosthenes [164] locate several Hellenistic monarchs from the third century B.C. in the same group of marginal amateurs.

If all advances of mathematical knowledge took place in such semi-professional circles; if advances really took place (and

they did); and if mathematics teaching in the liberal education even in the schools of philosophy was nothing but a faint afterglow; then we can feel sure that even the conceptual organization of Hellenistic mathematics was not affected by what went on in the liberal education. The only role of the latter, was an indirect one, namely to provide a base of recruitment for the circle of mathematicians and serious amateurs by showing young people that mathematics existed, and that it might offer interesting and serious entertainment to those who liked it.

There seems, it is true, to exist an inner and substantial link between liberal mathematics teaching and the progress of mathematical knowledge. But the influence seems to be one-way and to be exercised by progress on teaching. Indeed, it seems to be an educated guess that the important advance in mathematical knowledge during the fifth and early fourth centuries B.C. was important among the possible factors giving to mathematics such a prestige that it came to be regarded as a necessary part of general culture by many educators [165]. Supporting evidence comes from the composition of the mathematical curriculum which went into the paideia: It consisted of just those disciplines which were developed up to the early fourth century.

After having swept away the various levels of the liberal education we may ask whether not at least the centers of higher learning were carriers of a teaching affecting the development and conceptual organization of mathematics? After all, they were just as legitimate successors to the Pythagorean brotherhood and other pre-Socratic mathematical philosophers (those environments where teaching affecting the development of mathematical thought may have taken place according to the above discussions) as was the system of liberal education.

First the difficulty shall be mentioned that the centers of higher learning were largely identical with the schools of philosophy which constituted the highest level of liberal education. Still, nothing prohibits that activities of different sorts could take place inside the same institution. The same Academy which Plato located close to a gymnasium where young people congregated [166] (and where he practiced discovery learning in the teaching of junior pupils [167], in agreement with the method recommended in the Laws [168]) is the place where Plato and a number of other philosophers "lived together ..., making their inquiries in common" [169]. So, it seems legitimate to treat the centers of higher learning as separate institutions, in spite of their connections even to the highest level of liberal education of Ancient gentlemen.

The oldest of these institutions was Plato's just-mentioned Academy. Eudoxus, who disputes with Archimedes the place of honour of Greek mathematics, worked here, and so did according to Proclus [170] a number of mathematicians contemporary with the elder Plato.

According to Proclus' text [171] a number of these mathematicians linked with the academy participated actively in the methodological refinement of mathematics which in the end gave rise to the axiomatic construction known from Euclid's Elements. Some of them were also themselves centers of circles of mathematicians, "schools" [172], active in methodological discussions.

Although Plato himself was no mathematician the opinion has been expressed by several authors [173] that he was effectively the midwife of the transformation of the mathematics. Zeuthen would even speak of the fourth century transformation as the "Platonic reform". However, even if this is true the main influence should presumably be ascribed to his philosophical viewpoints. Plato's personal midwife style [174] and the particular organization of the circle of learning at the Academy and descending mathematical "schools" may have furthered productivity and creativity, but it is far from evident that they should have acted as social determinants of the direction of the creative process.

Truly, the circle of mathematicians centered on the Academy continued the tradition of Elements, which as already said looks like being dependent on teaching, even if perhaps only on weakly institutionalized personal master-student relations. This would of course even be a possible factor in the social context of Academy learning. But what may once have been started as a process determined by social factors was by the time of Plato and Aristotle already a process directed by conscious search for the target of systematization and rigour - targets ultimately derived perhaps from the conscious recognition of what already went on, but none the less established on their own by the mid-fourth century [175]. So, the recurrent re-editing of the Elements can just as well have been part of the impetus of mathematical activity (one could almost say "belong to established routine") as a result of a need for continuously improved textbooks created by a teaching network. In fact, we cannot sort out the influence of the respective factor: Philosophy, master-student-networks, mathematicians' standards and inherent impetus.

The Academy was only closed in A.D. 529, and its mathematical interests stayed alive until the very end [176]. But apart the ambiguous fourth century evidence just discussed nothing indicates that it ever played a role in the formulation or development of Greek mathematics. Its mathematical interest was on a higher level than that of the liberal education, but it was still secondary.

Aristotle's Lyceum [177] offers the same picture apart some details. It was never the host of a mathematician of Eudoxean qualities, but on the other hand the Aristotelian refinement of logical analysis was maybe not without influence on the logical refinement of mathematics. Still, this had nothing to do with the organization of Lyceum teaching, and the further existence of the Lyceum presents the same picture as the Academy'albeit Aristotle's lesser emphasis on mathematics and a concentration on exegesis of the Aristotelian corpus [178] made the Lyceum tradition less mathematical than that of the Academy.

The Epicurean and Stoic schools are apparently irrelevant to our subject. So are also the various neo-Platonist schools which are no different from the late Academy, at least from our point of view [179]. The Museum of Alexandria is then the only remaining important institution of higher learning. Clearly it played a central role for Hellenistic mathematics: "von rund 300 bis 100 v.u.Z. haben sich alle bedeutenden Naturwissenschaftler und Mathematiker entweder in Alexandria aufgehalten oder wenigstens vorübergehend dort gearbeitet und sind dann im Briefwechsel mit Alexandria geblieben" [180]. Everything indicates that Alexandria and its state-supported center of learning was the institutional root of the community

of professional and semi-professional mathematicians (cf. p. 36f), and thus of the culmination of Greek mathematics taken as a whole. As late as c. A.D. 300 Pappus worked in Alexandria [181], and even around A.D. 400 mathematically valuable commentaries to earlier mathematicians were written by Theon of Alexandria (the last member of the Museum known by name) and his daughter Hypatia (the head of the neo-Platonic school of Alexandria) [182].

Still, this does not point to a decisive influence from the fact of teaching. Only the Euclidean Elements, written presumably in the early epoch of the Museum and according to tradition at least under the same royal protection as that given to the Museum [183], may be an exception in two respects:

First in their creation. Euclid's Elements constitutes the final and the high point in the chain of such works ordering fundamental mathematical knowledge. The possible didactical roots of this tradition were already discussed.

Second in their use. Indeed, post-Euclidean mathematics is not only on the same level of axiomatic rationalism as Euclid's work. It is "Euclidized" to such an extent that it deserves close investigation when a piece of Hellenistic mathematics is built on a different foundation [184]. Euclid's Elements became the book in a way typical of a didactical institutionalization of knowledge - it became a paradigm in Kuhn's original sense of that term [185] - an exemplar studied by everybody practising the discipline, thus securing a basic uniformity of thought and expression.

On the whole, however, the distinctive character of Greek mathematics was already fixed by the creation of the Museum and so before Euclid wrote his Elements. Apart the final Euclidization, therefore, neither Museum nor any other institution of higher learning influenced the structure, organization or style of mathematics in the Hellenistic time. And so, didactical organization seems only to have exercised a possible influence on the formation of Greek mathematics in the pre-Platonic period and (less probably) the Platonic and immediate post-Platonic Academy.

The heirs: India and Islam

Greek mathematical knowledge was transmitted to later times through several channels. One of these is Byzantium, where no significant further work or reformulation of mathematics took place. So, the custodian tradition of Byzantium is irrelevant to our subject. The other main channels descend from early Medieval Syriac and Pehlevi learning on which little is known but which seems not to have offered much more than Byzantium as far as new developments are concerned [186]. For both reasons I shall omit them from the exposition, and jump directly to the "great" traditions of Medieval mathematics: India and Islam [187].

On Indian mathematics I shall say very little, because of my far-ranging ignorance. There seems to exist from early (Vedic?) times an independent mathematical tradition, containing both geometrical constructions [188] and play with huge numbers [189]. The followers of the Jaina religion, which from its creation in the sixth century B.C. exhibited a great interest in mathematics

and astronomy [190], were probably well advanced in independent studies of algebra and combinatorics around the first century B.C.

By Hindus as well as Buddhists and Jainists the mathematical tradition was dependent on religious cults [191], and it was used by priests in their ritual function [192]. Already in the second millenium B.C., an education system was established where everybody was expected to learn his family craft and the Vedic hymns, the latter by extended memorization. This education took place inside the family [193]. During the first millenium B.C., the social classes hardened into castes, the priestly caste taking the absolute lead [194]. The sacrificial rituals became increasingly complicated (by a process presumably parallel with the cultural refinements introduced by Babylonian scribes in their craft); religious instruction became the privilege of the brahmin priests, and it was extended to twelve years of study, increasing immensely the amount of necessary memorization [196]. It was in this context of Brahmanic ritual and Brahmanic education in the sacred books that the first texts on ritual geometry were created. This origin is clearly reflected in their style: Rules, formulated in a cryptic style and separate stanzas, to be explained by the teacher and learned by heart.

Indian education continued its dependence on memorization for very long. In the first millenium A.D. books on many subjects were written in metrical form, including not only mathematics but even dictionaries [197]. Even in the beginning of the 20th century, the brahmin boy was taught his arithmetic in verse (as appears from B. Banerji's novel Pather Panchali [198]).

In later times, learned commentaries were written to the treatises of ritual geometry, some of them apparently by commentators versed in Greek geometry [199]. In the first centuries A.D., Greeks imported Babylonian as well as genuine Greek astronomy into India, where it was absorbed in native astronomy and brought it to a much higher level [200]. This was the occasion for really advanced developments of mathematics. None the less, Indian mathematics retained its peculiar character throughout the Middle Ages, different from that of Babylonian and Egyptian as well as from that of Greek mathematics. The main and ever recurring element of an Indian mathematical text is the versified rule, given in a laconic and general formulation but without neither proof nor reason. This rule may be followed by one or more examples showing the application [201]. The examples may then be followed by a proof that the result found fulfils the conditions of the problem [202] (i.e. not a demonstration of the general validity of the rule).

I would guess that this conservation of style in spite of great infusions of foreign knowledge and impressing theoretical development must be ascribed to the maintenance of the original organization of teaching around memorization - connected, of course, to the way the mathematics in question was used, not only throughout the Middle Ages but still in 1825, when an English scholar "found a 'Kalendar maker residing in Pondicherry' who showed him how to compute a lunar eclipse by means of shells, placed on the ground, and from tables memorized 'by means of certain artificial words and syllables'" [203]. One is reminded of the Chinese Buddhist pilgrim who in c. 675 told about an Indian school which he had visited, where "certain aids to memory (were) used; after a period of 10 to

15 days' practice the student was able of committing a work to memory on one hearing only" [204]. No wonder that mathematics exercised under such conditions became different from that of the clay tablets, the papyrus and the discussions around the lines drawn in the sand.

Even on the world of Islam I shall not say very much. The reason for this is not that my knowledge is quite as scarce when Islam is concerned as in the case of India, but rather that fuller explanations would require a thorough discussion of Islamic history, society and culture, and especially of the religious and political contradictions of Islam [205].

Three main branches of Islamic mathematics [206] can be distinguished, even though, of course, they are mutually connected and often worked on by the same persons.

The first contains practical arithmetic, algebra and "geometrical practice" (what in the Medieval West was called practica geometriae; in other words Heronic geometry), extended with elements of Pythagorean number theory (especially summation of series).

The second is the "mathematics of astronomy", trigonometry, including spherical trigonometry, numerical techniques, the algebra of approximate solutions, and maybe that of indeterminate equations.

The third may be called "pure mathematics", dealing especially with extensions of Euclidean geometry, but including also questions of higher algebra [207].

The first branch can be considered the "mathematics of social life". It is more or less split in two interdependent traditions: The extended arithmetical textbook, and the "algebra with or without geometrical practice".

Strictly speaking, the arithmetical textbooks are of two sorts, one is the "algorism", the description of the use of Hindu numerals. Those which we know are not visibly connected with teaching, and so I shall omit them. The other sort consists of the genuine "Rechenbücher", which may use Hindu numerals, the pre-Hindu "finger-reckoning" rooted in Near Eastern traditions (Greek, Arab, and maybe Egyptian), or both. They extend from the middle of the 10th to the middle of the 15th century [208]. A number of these books can be connected to various sorts of schools, and at least from the 11th century onwards they must all presumably derive directly or indirectly from a common trunk: The teaching of the Madrasa, the higher, primarily religious, school.

The reason why a considerable amount of mathematics (be it practical mathematics) was taught in a religious school must on the level of religious doctrine be seen in the fact that Islam as a religion claims to be able to answer all problems of human existence - including the problems of practical life. On the sociological level one will notice that a significant majority of Muslim religious scholars belonged in their secular life to artisans' and merchants' occupations [209]. So, there was no social basis for a separation of secular concerns and religious teachings. Furthermore, since the Muslim world failed to develop urban autonomy on the lines of Medieval Western Europe [210], no pressure to develop such a social basis was present.

It should be noted that the Rechenbücher normally contain a section on the summation of series (n, $2n$, $2n+1$, n^2, $\frac{1}{2}n(n+1)$, n^3, 2^n). Historically, this seems to have been taken over together with the finger-reckoning, in the same breadth and from the same sources [211]. The reason that it survived in a practically oriented tradition must, however, be sought separately. It may perhaps have to do with the esoteric, gnostic, neo-Platonic and mystical currents which were always present in Islam, and which were at least interested in another game with abstract numbers: That of magic squares [212]. However, at least part of the esoteric environment interested in magical squares (the ṣūfī's) were from c. 800 most scornful of commerce [213]. So, a better explanation would probably be an ascription to the dynamics of mathematics teaching, conserving and extending the subject through time because it could be taught and extended theoretically (there is a tendency that the subject is treated more fully the later the treatise).

However, even if the interest in series and their summation should be ascribed to the didactical rooting of the Rechenbuchtradition, another influence of the Madrasa teaching tradition was much more important. I think of the gradual suppression of the traditional finger-reckoning and the advance of Hindu methods. There is some evidence that this advance of a more systematic notation (a notation which permits theoretically better discussions of methods) inside the domain of mathematics for practitioners took place in spite of the practitioners' tacit opposition [214] - at least the sort of factual opposition consisting in the practitioners' inability to adopt a new, high-level practice if not propagated by a sufficiently institutionalized teaching system. Corresponingly, it seems that the more satisfactory Indian notation for general fractions (instead of products of the "natural fractions" of the Arabic language, 1/2, 1/3, 1/4, ... 1/10, and their complements 2/2, 3/4, ... 9/10) was advanced not by its practical usefulness but rather through the teaching of the school [215].

Another aspect of the Rechenbücher deserves to be mentioned. In the later ones, incipient mathematical symbolism can be found [216]. This must presumably also be put on the account of the dynamics of teaching. Since, however, this development took place in the Indian summer of Islamic mathematics (the 12th century onwards), it remained a beginning without consequences as long as we restrict our interest to the Islamic region. On the other hand, it was presumably the source of the corresponding introduction of pristine symbolism in Western European mathematics in the 13th century by Leonardo Fibonacci and Jordanus Nemorarius [217]. So, the development may have been most influential on the long run of mathematical development.

Still, symbolism (or proto-sybolism) was not the main heritage of the Madrasa school and its relatives. By far more important was its propagation of the Hindu numerals, which were also spread through the Latin West from the 12th century onwards on the basis of texts translated from Arabic.

Most of the Rechenbücher contain a section on algebra - just as does the Liber abaci of Fibonacci. Besides, however, a number of treatises of algebra without arithmetic, and with or without geometrical practice, can be found. I have not investigated these well enough to discuss didactical influences on

the development of this tradition. On the face of them I would presume that such an influence is rather modest. However, the origin of the tradition may depend on teaching: "Revenant à l'algèbre d'al-Khwarizmi nous dirons qu'elle appartient à un courant didactique qui a nourri antérieuerement l'oeuvre de Diophante" [218]. This didactical current should, according to Anbouba, be of Babylonian origin, a guess I would strongly support e.g. on the evidence of common numerical examples [219].

The mathematics of astronomy became important because astronomy was so, and especially because astronomy was exercised as a science in progress. The interest in astronomy had several causes. One is the ritual of Islam, which makes it necessary to find the hours of the prayers and the direction towards Mekka. Another one is the connection to geography in general. The most important cause, however, is probably to be sought in princely interests in astrology, which permitted the funding of observatories and of astronomers, the prime task of whom it was to construct zīj, i.e. astronomical tables, usually introduced by chapters on theoretical astronomy [220]. It was this last motivation, we must presume, connected with the existence of a tradition recognizing the possibility of a cumulative science, and with a princely wish to be glorified, which made Islamic astronomy a science in progress.

Of course, astronomy was taught, and so the mathematics of astronomy. The directing principles of this teaching, however, lay in the application, and there is therefore not much influence from the facts of didactics to be expected on Islamic astronomical mathematics. Furthermore, astronomers considered as a body of practitioners were compared to merchants' clerks or officials, much less in need of an institutionalized school to systematize and disseminate their mathematical techniques. So, even though astronomy and its ancillary disciplines were in fact taught according to organized and partly institutionalized schemes, the branches of mathematics used specifically in theoretical astronomy were not affected in their development by the education through which they were transmitted.

This does not imply, however, that the astronomers' education was without decisive influence on Islamic mathematics. In fact, I will maintain that the existence of its third branch, that of "pure mathematics", is strongly dependent on this education.

First it should be noticed that the immense majority of the mathematicians of the Islamic world were also active as authors of texts on astronomy. Most of those who are not known to have written on astronomy are known exclusively for work inside the arithmetico-algebraic branch of mathematics. Such a close connection of personnel between the two disciplines makes a total absence of cognitive interaction implausible, but of course proves nothing. Neither does it point out the specific fields of possible interaction.

Now, a pointer is provided by a term used to designate a body of mathematical and astronomical works of mostly Greek origin. Euclid's Data, Optics and Phenomena; the pseudo-Euclidean Catoptrics [221]; Theodosius' Spherics; Autolycus' Sphera mota; Archimedes' On Sphere and Cylinder, Measurement of the Circle, and Lemmata; further works by Apollonius,

Theon of Alexandria, Pappus, Hero, Hipparchus, Diophantus, Nicomachus, al-Khwarizmi and others; all these are referred to as the "middle books" [222] because they were studied between the Elements and Ptolemy's Almagest, obviously in the astronomers' education. Evidently, even if fixed and far from all books studied in all cases [223], this curriculum is mathematically ambitious. It cannot all possibly have been taught just as leading forward to astronomical practice; instead, the mathematical part of the curriculum in question (Elements and the mathematical parts of the middle books) must be considered a rather autonomous mathematics curriculum[224]. In any case, the Western medieval heirs of the Islamic astronomical tradition would easily do without this curriculum and replace it with rather simple compendiae. So, its wide scope and high level was no necessity for the exercise of astronomy: its raison d'etre must thus be sought for elsewhere.

There exists a large body of commentaries to the books belonging to this fundamental mathematical curriculum; seemingly growing out exactly from its being a curriculum, the object of systematic teaching. Many of these commentaries belong to the category of pure mathematics - and even to research in the foundations of mathematics [225].

It would be hazardous to suggest that all Islamic pure mathematics was made in direct connection with the teaching of the basis of astronomy - there is no evidence it was [226], rather some counter-evidence. But it is plausible that much of it was connected with teaching [227]; and further, that the activities centred on teaching made socially visible the possibility of working on pure mathematics.

Latin Middle Ages: The primacy of teaching.

It was asserted above that Greek mathematical knowledge was transmitted to later times through Byzantium, India and Islam. When the word knowledge is emphasized this is near to complete truth. The Greek estimation of mathematics as an important part of human learning was, however, transmitted directly to Latin Medieval culture - paradoxical as this may seem [228].

True enough, the break-down of the Westerm Empire led to an unsuspicioned decay of the level of all branches of learning, and especially of the level of mathematics. For about 400 years only one mathematical demonstration in the Greek sense of that concept was studied in the West (if it was studied at all), one belonging to Boethius' Latin version of Nikomachus' compendium on harmonics [229] (and ultimately going back to Archytas) - in so far as this rather complicated treatise was studied at all. Still, even in the darkest Middle Ages a rudimentary school, descending primarily from the Ancient liberal education, continued to exist at the episcopal households and, less formally organized perhaps, in connection with the Benedictine monasteries. Outside the church, almost all schools vanished, the only exception being private teaching of rhetorics in Italy. All secular learning vanished too.

The decay of learning was caused mainly by social factors: The breakdown of the Roman state, the virtual disappearance of urban trade, urban crafts and urban culture. But it was also sort of accomplishment of ecclesiastical wishes: The

attitude of the young Christian church towards heathen learning had been ambiguous but rather critical and suspicious. The more remarkable is it that the vestiges of schooling and learning connected with the early medieval church carried over - not so much mathematical knowledge as the knowledge that mathematics existed; that it consisted in the four disciplines of the quadrivium concerned with numbers, figures, the movements of the heavens, and musical harmony; and that mathematics, this virtually unknown subject, was an important field of knowledge: "We are not confounded but instructed by number. Take number away from things, and everything perishes" [230].

The bishop's "family school" and the early Medieval monastery were mean places of hibernation. Still, they were better than none. Even if low, their level was not equally low everywhere, and the episcopal and monastic learning were to become the basis for the late 8th century "Carolingian Renaissance". Starting from this, in a retroactive process, monastic schooling and learning were slowly and irregularly but yet decisively improved, and genuine cathedral schools were established in the tenth century.

Neither the users of the ecclesiastical learning nor anybody else in the Western Central Middle Ages had any material need for advanced mathematics. Finger-reckoning, used in the "computus" (Easter reckoning and calendar calculation in general) and perhaps elementary accounting; the use of the abacus (which was rediscovered or, rather, imported via some channel from the Islamic world around 980 [231]); and the bit of surveying techniques used by architects of monasteries and similar buildings. These bits covered all practical needs for mathematics. Even in the High Middle Ages, c. 1100 - 1300, the practical needs for mathematics were restricted to the above, combined with the use of Hindu numerals, somewhat better accounting methods, and practical geometry including surveying (as long as we do not consider as practical the requirements created by astronomy and astrology). So, practical needs could hardly be the reason for a revival of mathematical interests in the Central Middle Ages.

Yet such a revival did result from the Carolingian Renaissance. Just as it happened to manuscripts on logic, mathematical manuscripts were dug out of the monastic archives, copied and transmitted via the school. This happened to a fragment of Boethius' 6th-century Latin translation of all or part of the Elements; to various Roman geodetic writings which by their Central Medieval diffusion have created a (presumably false) impression that monasteries had a general practical need for surveying [232]; and to Boethius' extended translation of Nicomachus' Introduction to Arithmetic.

One may wonder why such an interest was aroused. As far as I can see, the only explanation is to be found in the dependence of ecclesiastical schooling and learning on the liberal education, and in the admiration of the literate for the culture of Antiquity (which, after all, was the ultimate and often the direct source of most of the material used in teaching, be it grammar, rhetoric, logic, theology or arithmetic). Good teaching was understood as comprising the quadrivial disciplines; so, arithmetic and geometry had to be taught; this provided opportunity for the forgotten manuscripts to be integrated in the living teaching tradition when

they were discovered - and when only surveyors' manuals were available, these had to be used as the basic geometric texts - as no one was competent to reconstruct basic theoretical geometry from the hints of its contained in encyclopedic texts from late Antiquity and the earliest Middle Ages and in Ancient Roman geodetic manuals. This is amply demonstrated by various correspondences on mathematical questions from the late 10th and the 11th centuries written by cathedral school teachers and their former students [233].

Briefly stated, the Central Medieval need for basic theoretical mathematics was a cultural need, created and upheld through the activity of the school. It was just one of many expressions of the longing for a renaissance of Ancient greatness which penetrated the educated environment throughout the Middle Ages.

The economic revival of the late 10th and following centuries; the coming into existence of organized lay and ecclesiastical administrations; the reappearance of trade and the growth of towns and artisans' industry; the growth even of royal and noble wealth; - all these created the native background for a cultural bloom. This bloom was far from totally bound to church, school or antiquity - just think of epic writing and troubadour poetry. Yet as far as learning is concerned the bloom was strongly connected to the cathedral schools of the 11th century. They were the main factor behind the endeavour to extend the knowledge at hand, both in direction of what had been known to the Ancients and in direction of the Muslims and Jews of the Islamic world. Truly, the 12th and early 13th century translations of Arabic writings were not a product of work going on inside the schools but rather of single devoted scholars who settled in the border regions. But the background of these scholars was that of the schools, and their translatory productions went mainly into the tradition of the schools (a few exceptions there were of course - the Salerno school of medicine and the court of Frederick II in Naples). This is especially true of the mathematical translations, where the 12th century reception of Euclid's Elements and of Ptolemaic astronomy (and astrology, its senior partner) are most conspicuous, but where algorisms and treatises on algebra should not be forgotten.

In the twelfth century, the development of learning was still in an undecided state. The main current was directly descended from the earlier Western tradition, supplemented to a certain extent with translations from Arabic of Ancient works which had previously been missing (e.g. part of Aristotle's logical works). Minor currents were, however, strongly dependent on the imported disciplines. So, in the late 12th century one ecclesiastical polemicist complained about students loosing their soul by the intensive study of Euclid, Ptolemy and other Ancient authors, a study done by these students as philosophy and not as part of the legitimate liberal arts [234]. There may indeed have been reasons to mention Euclid as one of the dangerous philosophers - one of the several translations of the Elements made during the 12th century was apparently widespread, to judge from the number of conserved 12th century manuscripts.

That exactly this version was spread must be interpreted as the outcome of a process of natural selection acting upon the total population of versions of the work (six in all, of which the one made from the Greek has been described as "the most exact translation ever made of the Elements" [235]). The version which became most popular was indeed no mere translation; it was rather a didactical commentary (a commentum, in the authors own words) on the series of Euclidean propositions. It is obviously made for and fit for a school not (not yet) ready for pure, unexplained and rigorous deductivism - and in agreement with Darwinian principles the survivor was the version best fit for this natural habitat.

The didactical commentary did not keep inside the Euclidean framework. It made cross-reference to other parts of the quadrivium, and it shows some affinity to the teaching of grammar. Even when the version in question was reworked in the mid-13th century and provided with rigorous proofs (this version due to Campanus of Novara being the final "Medieval Euclid"), the imprint of a broader school environment was conserved - the didactical commentary is still there between the propositions, and references to Plato, Boethius and Aristotle are found along with numerical examples supporting understanding [236]. Medieval Euclidean geometry never became an isolated pure science as it had been in antiquity (and even in the Islamic pure mathematics preparation for astronomy). It was presented as part of more all-embracing entities, the quadrivium, the liberal arts, and the philosophical teaching of the schools in general. So, the rooting of pure geometry in the school environment was a factor of unity and coherence - but a unity embracing larger entities, thereby dissolving to some extent the internal unity of mathematics, as it appears.

Around the end of the 12th century, some of the schools (or, in the case of Paris, clusters of independent schools and single teachers licensed by the cathedral) had grown so big that the scholars (masters and students, or students alone) organized themselves in guilds - in their professional Latin tongue: in universities. The ensuing struggles for the rights of these learned guilds for independence from ecclesiastical authorities was one factor favouring the creation of a uniform curriculum. Another (synergetic) factor was the reception of the complete Aristotelian system, including metaphysics, natural philosophy and moral philosophy. This system was so far superior to all intellectual competitors (including Euclid and Ptolemy) that the whole basic training in the arts faculty was united under the common hat of Aristotelian logic and natural philosophy. This was of enormous consequence because of the structure of the university education. Every student first had to spend seven years in the arts faculty (from c. 14 to c. 21) unless part of this time was replaced by corresponding teaching in other institutions. Here the liberal arts were taught, increasingly biased toward dialectics, and increasingly supplemented by natural philosophy. For most students, the arts faculty was their only university training. The rest who continued their studies in higher faculties (law, medicine, theology) would normally make a living by teaching at the arts faculty while continuing their studies. So, what went on in the arts faculty meant everything to all learning in the 13th and 14th centuries.

The primacy of dialectics and natural philosophy did not imply that mathematics was no longer treated. Indeed, the universities were the place from where Hindu numerals were primarily spread (at least outside Italy). The process by which this happened is

worth mentioning. The first algorisms translated never spread very much. Early in the 13th century, a versified algorism became quite popular in the universities. This collection of rules without arguments was soon replaced or supplemented inside the university by another more arguing and explaining work by one John of Sacrobosco. A late 13th century commentary to Sacrobosco's algorism is still more rigorous, and - characteristic of the time and environment - before setting out to discuss mathematical details it explains the four Aristotelian causes of the work [237] (the effective, the material, the formal and the final cause). Concomitantly, the art of algorism began to spread outside the university, in vernacular versions, of which quite many manuscripts from the 13th, 14th and 15th centuries exist. Most of these are versified, translations of the original Song of Algorism from the early 13th century university. Lay circles had no use for explanatory expositions, and a fortiori not for philosophical distinctions. These are specific products of the university environment, and absent in the vernacular algorisms.

Even Euclid was studied to some unknown extent in the universities - according to the irascible Roger Bacon hardly 3 or 4 propositions [238], according to an anonymous compendium from Paris university from the 1230'es all 15 books [239]. In view of various sources for the total syllabus [240] and ecclesiastical complaints [241] it seems that some students but by no means all read at least several books - six being the norm in Oxford.

From the mid-13th century onwards, the Euclid read in the universities was the Campanus-version mentioned above. Its relation to its most popular 12th-century predecessor was already presented: It is logically rigorous, and establishes connections not only to quadrivial arithmetic but even to Platonic and Aristotelian philosophy. Even though Campanus was probably holding ecclesiastical office and no longer a university man when he made it, his version is clearly written as a contribution to a mathematical discipline conceptually moulded by the university institution, and especially by the teaching of the arts faculty. As the type of learning of the Medieval university and especially the arts faculty has got the name <u>scholasticism</u>, we may legitimately say that Campanus' Euclid was the best piece of scholastic mathematics of the 13th century.

These words were chosen with particular care. Firstly, Campanus' Elements form a typical scholastic work: A commentary on a pre-existing piece of learning (keeping close but not abnormally close to the original), containing many fine points but no revolutionary thoughts - as natural inside a teaching tradition built on authorities but aiming at the critical understanding necessary when the often conflicting authorities were to be subjected to the renowned scholastic disputation. Secondly, Campanus' work was <u>not</u> the best piece of Latin 13th-century <u>mathematics without an epithet</u>. Two writers of the early 13th century, Leonardo Fibonacci and Jordanus Nemorarius, had written works of much greater mathematical originality and competence. These two mathematicians were not unknown to their contemporaries and to the next generations of university mathematicians. A reasoned algorism containing demonstrations as well as mathematical generalizations by a follower of Jordanus [242] even became quite popular at the universities, as it was in better agreement with their general spirit than the accumulation of mere rules with slight comment. But in general

their work inspired no continuation, and the university retained what Suter [243] has called "die dem Mittelalter eigene, vorwiegend philosophische Auffassung" of mathematics for many centuries.

Instead of cumulative conceptual expansion of mathematics, the late 13th and the following centuries favoured the use of mathematical <u>compendia</u>, a clear demonstration that mathematics had been reduced from the status of the high point of general culture (as the quadrivium culmination of the liberal arts) to that of a body of <u>ancillary disciplines</u>. True enough, the intellectual level as a whole had been so much raised and even ancillary disciplines had been so intensely submitted to the general didactical and philosophical standards that late 13th century compendia were more rigorous, of a wider range with regard to subjects covered, and conceptually more rich than the quadrivium texts of 1100. Still, ancillary sciences which have no autonomous activity tend to stagnate, and in fact the compendia of the thirteenth century were printed in great number in the 15th and 16th centuries, being still in general use.

I have made some scattered remarks on the harmony between scholastic mathematics and the teaching of the schools and the university. We may approach the question in a more systematic way and ask from the point of view of educational sociology: Why was mathematics just an integrated part in a philosophically oriented whole, and why was it eventually reduced to an ancillary status? Couldn't the Western Medieval school and university have achieved instead what was achieved in the Islamic world, where mathematics retained and improved its autonomy in spite of its binding to astronomy?

Probably there were institutional reasons that it could not (institutional reasons which of course had their reasons, the discussion of which I shall omit). The main carriers of general knowledge with no specialized occupational direction were the arts faculties (and the friars' orders which had penetrated the university environment and consciously imitated it). Only at this level of the university would mathematics have a place; neither theology nor law or medicine needed it (I disregard Bologna medicine which taught mathematics for the sake of astrology which was taught as a tool for medical prognostication). In the arts faculty you studied until an age of c. 21; eventually you might become a teacher in the same place for a decade or less while studying at a higher faculty. As a master you normally taught varying subjects, just as you had learned all subjects while an arts student - and there was even a tendency that you taught and learned mathematics mostly on holidays as an optional activity. This did not open the way for the formation of a group of professional mathematicians, not even in the loosest possible sense of these words. The very few people to whom mathematics was a "way of life" (as Campanus of Novara, if we include astronomy, and maybe Jordanus and Leonardo Fibonacci) remained isolated cases; there was therefore no social basis for a steady continuation of their work. At most, we may regard the masters of art as a body of temporary professionals practicing the arts as a totality. This quasi-profession - which included university mathematicians as a non-autonomous and a non-identifiable subgroup without permanency - did create a collective cognitive expression: the scholastic style and philosophy - which even became the way university mathematics was expressed.

Curiously enough, while still using the same invariable compendia for general quadrivial mathematics the 14th century witnessed new peculiar mathematical developments created specifically as an extension of scholastic philosophy by the environment connected to the teaching of arts (masters of the arts faculty, or former masters who in their later career continued the same sort of scholarly activity). Refined discussions of Aristotelian philosophy, especially of the Physica, fertilized perhaps by Galenist medical philosophy, led to investigations of the concepts of graduation, speed, uniformity of speed, uniform and "difform" acceleration, and to investigation of ratios and their composition. This has been regarded as scholastic mathematical physics. Even if we employ the term "physics" in its Medieval sense of "philosophy of nature" this is an unhappily chosen term. A more adequate term is that of "mathematical philosophy" [244].

This application of mathematics in philosophical argumentation soon became an autonomous branch of philosophy, without necessary binds to its applicability in philosophy in general. It became sort of pure mathematics [245], that is, the specific pure mathematics of scholastic philosophy. Due to its relation to scholastic philosophy in general this sort of mathematics could become the main activity of whole groups of people in certain university localizations which acted as foci, especially Merton College in Oxford. So, there were interconnected philosophical and sociological reasons that mathematical philosophy did not suffer the general fate of quadrivial mathematics, and that it was not regarded as an ancillary discipline but rather as a legitimate interest for "professional" masters of arts. It had its own compendia late in the 14th century. Ultimately when the final ossification of the scholastic university occurred its development stopped. The young Galileo may still have been taught by means of the compendia of the 14th century.

An illuminating parallel can be found in the 14th century development of logic. This science of argumentation, truth and falseness was refined to a point where any connection to everyday argumentation was cut off - and even the umbilical cord to normal university disputation may have been strained. Socially regarded, this 14th century "modern logic" (a term distinguishing it from the "Ancient" Aristotelian logic) became a kind of pure mathematics. To its contemporaries, it belonged of course to another realm of knowledge, that of philosophy, just as did mathematical philosophy. Even with regard to the type of social rooting in the semi-profession of the masters of arts, the two disciplines were more or less in the same situation. The different fates of quadrivial mathematics and mathematical philosophy, taken together with the almost identical fortunes of "modern logic" and mathematical philosophy (and, of course, of their common character of freshly developing art pour l'art) supports the idea that we have to do with two phenomena of the same sort, and thus also the combined philosophico-sociological explanation of the development of autonomous mathematical philosophy proposed above.

This is, in breadth but definitely not in depth, most of what there is to say to our subject in connection with the Latin university-based mathematics of the Western Middle Ages. The most urgent want is a closer investigation of the connection between the organization of university teaching (lectures,

disputations etc.) and the structure and character of scholastic thought and learning in general - a connection which was left as much of a postulate.

However, even if we leave the Medieval university we are not through with Western Medieval mathematics in general. Truly, the treatises of geometrical practice (which in part descend from the 12th century school and in part probably not, and which were not mentioned) seem as far as I can see not interesting in connection with our subject. But a third tradition inside Western Medieval mathematics can be traced from the 13th century onwards, even if the evidence is quite scattered [246]. This tradition was probably neither quite isolated from nor inimical to the university tradition; but it was still a tradition of its own with a social basis of its own. It was socially connected with the commercial environment of the Italian cities and based probably in the so-called abacus-school. It drew, from Leonardo Fibonacci's Liber abaci onwards, on Islamic treatises of the Rechenbuch and algebra traditions, of which it was a full and direct continuation. Its interests were not restricted to the commercially useful, as far-fetched instances of higher-degree algebra (up to the sixth degree) occur [247]. As in the case of Old Babylonian higher algebra these equations of a higher degree provide us mainly with a proof that the maestri d'abaco did not master such problems. The general rules offered are wrong, but most of them work in special cases. They may have permitted the maestro (or the good student) to exhibit his skill, and seem thus to be there by virtue of the specific social belonging and organization of this tradition (this explanation is supported by the circumstance that wrong prescriptions were transmitted from one treatise to the other).

Islamic algebra had been translated already in the 12th century by those same scholars who translated the bulk of mathematical works going into the university tradition. But algebra had on the whole been neglected inside the university tradition. So, the virtuoso's interests of the maestri d'abaci (and the social make-up and rooting of the abacus school which was the raison-d'etre of these interests) was not only responsible for introducing and keeping alive a number of errors with no practical bearing, taught in the abacus school. In fact, they were the main responsibles for the transmission of Islamic (and so, ultimately, Babylonian) algebra to the early renaissance.

In the 15th century, the scholastic university was hit by intellectual sclerosis - a result of the breakdown of the "feudal synthesis" headed by the church, of the disappearance of university autonomy and of the disappearance of ecclesiastical affluence. The arithmetico-algebraic tradition, however, survived the crisis as did the mercantile interests, as did the abacus schools for merchants' sons and clerks, and as did not least the cultural expression of bourgeois patriciate and princes: Renaissance culture. It was even in the 15th century that it began to flourish, and in the end of that century that Luca Pacioli (writing extensively inside this tradition) recognized that equations of the third degree could not be solved [248]. Shortly before Nicholas Chuquet wrote his Triparty, a large arithmetico-algebraic treatise clearly influenced by the Italian tradition [249], demonstrating thereby that the tradition was reaching France. Already by the middle of the 15th century treatises of commercial arithmetic and simple algebra had begun to spread in Germany [250]. At this time, in the second half of the 15th century, the currents of university mathematics,

renaissance high-level mathematics, and the arithmetico-algebraic tradition had begun to interact quite strongly, but until the early 16th century we must still presume that the abacus school and its relatives were the main carriers of algebraic interests, and chief responsibles for its survival and flourishment. From then on, printing was beginning to cause radical change [251]. From the early 16th century, too, no further progress in the field was produced by the commercially rooted abacus school; instead, a research front of algebra was soon created by genuine scholars like Bombelli and Vieta. From then on, algebraic progress was disconnected from teaching practice, and mostly the product of independent scholars and authors.

This was a general feature of Renaissance mathematics. The ossified universities continued their teaching; there was even a slow (but very slow) renewal of the mathematical compendia in use, and the new developments of the Renaissance and even the Modern period penetrated to some extent the university curricula. But this penetration was a penetration of didactically simplified versions of theories and techniques already developed outside the universities. So, the innovations in late Renaissance and early Modern mathematics owed nothing, be it style or be it contents, to teaching in the university or in other institutions. Only around 1800 was this situation changed once more, under the impact of the creation of the École Polytechnique and the German university reforms.

Conclusion

Is a conclusion at all possible? Did the investigation produce anything but a caleidoscopic picture, changing at every turn of history?

In a certain sense the answer is *no*. No ever-recurring features of the relation between institutionalized teaching and the development and organization of mathematical thought turned up. The particular in history is indeed particular.

On the other hand, it is almost a classical experience of anthropology that *any* generalization is falsified by some marginal society. The network of natural and historical conditions, and of interrelationships between the multitude of social institutions is too involved for any simple generalization to hold true everywhere. So, there is no reason to be particularly disappointed.

Indeed, it did turn out that institutionalized teaching has in several historical situations been a strong factor in favour of the systematization and the rationalization of mathematical knowledge. It also turned out that the *type* of systematization etc. produced (Babylonian, Greek, Latin Medieval) depends on the structure and social foundation of the teaching institution involved. "Beweisen" may indeed be a didactical problem [252], but the contents of the *Beweisen*-concept possesses no transcendental permanency. It finally turned out that institutionalized teaching would not always provide any impetus towards increasing systematization neither of its own subject-matter nor of mathematical knowledge in general (Syria in the second millenium B.C.; India), and that systematization might also derive from other social sources than teaching. The case of Hellenistic mathematics provides one instance. Another one is

provided by the Renaissance establishment of the *internal* mathematical coherence dissolved into the more general philosophical coherence in the Medieval University.

So, institutionalized teaching will, so it seems, in general favour the systematizing tendencies in mathematics. But it will not always do so, and other forces may work to the same effect.

NOTES AND REFERENCES

1 The first to pose the question and give reasons for an affirmative answer was perhaps Judith Grabiner (1974, 36o).

2 I don't agree that "the belief that mathematics is unique has exactly the same status as the belief that there is a unique moral truth" (Bloor 1976, 94). Furthermore I would suggest that to the extent that this major premiss is true it may cast doubt on the implicit minor premiss: That moral is nothing but historical arbitrariness.

3 The formulation is taken over from Q. Gibson (198o, 28).

4 See, for instance, Struik (1948; 1967, 7-15). A useful even if not totally reliable supplement is Fettweis (1932).

Still older precursors of mathematical thought (but definitely only precursors, as far as I can judge) are described by Marshack (1972; 1972a; 1972b; 1976) and Couraud (198o).

5 This system, which is based on the use of small tokens of various shapes made of burnt clay, is described by Denise Schmandt-Besserat (1977; 1978).

6 Childe 1971, 1o1.

7 This conclusion can be drawn from material presented by Rottländer (1976, 49) and Wright (1975, 282).

8 Of course, similar and in all probability rather independent developments have taken place in other areas: Egypt, China, Vedic India (whose dependence on the Indus culture and thereby on connections to Sumer is unclear) and, less certainly known, Neolithic Britain (cf. MacKie 1977, and the bulk of writings on the British megalithic "observatories"), Mexico (Closs 1977a, 83-9o, 1o3-12o) and Peru (Closs 1977a, 91-1o2). All of these developments are, however, later than the early Sumerian mathematics and, furthermore, less or not at all participating in the development toward modern mathematics. So, in a double sense mathematics was first created in Sumer.

9 This exposition of the process leading forward to the formation of the proto-Sumerian state is a synthesis founded on many different kinds of evidence:

General theories of the process of early state formation, as presented by e.g. Mèlèkèchvili (1967 - a very important paper on the problem of the so-called "Asiatic mode of production" and on pre-capitalist formations in general); for an early approach along the same lines, cf. Engels, Anti-Dühring (MEW XX, 166f).

Evidence from the meager and largely undeciphered contemporary written sources (cf. Vaiman 1974; Tyumenev 1969, 72ff) as well as from later Sumerian sources (analyzed in the extensive secondary literatur - I shall go into no details).

Pre-Sumerian and proto-Sumerian archaeological evidence concerning the temples (CAH I[1], 333-339; Childe 1971, 99f), the irrigation systems (Lamberg-Karlovsky 1976, 62) and urban productive technology (Nissen 1974). Cf. even R. McC. Adams, who presents (1966, 44-51) an exposition based on ecologically oriented archaelogy and supported by Sumerian written sources.

Even if my synthesis of this material is private, it is not argue for the ways in which I have combined the evidence.

1o On the concept of an "urban revolution" and the new social role of the city: See Childe (195o; 1952) and Kraeling (196o). Huot (197o) discusses the difference between the city and the sometimes large, pre-urban, agricultural agglomeration from the point of view of internal organization and reaches similar conclusions concerning the novelty of early Sumerian developments.

11 The transformation was a process of several steps (see Schmandt-Besserat 1977, 25-27; 1979; 198o), including first a violent proliferation of the number of different material counters and symbolic tokens in use, followed first by simple impression of the tokens in clay tablets and then by drawings representing the tokens supplemented by genuine pictographs, drawn on the tablets. The whole process of transformation seems to be dependent on the growth of state-like structures and on the transformation of pre-urban, temple centered agglomerations into genuine towns. Strictly speaking, neither of the three processes is restricted to the Sumerian area; in fact, much of the evidence comes from neighbouring areas in Southwestern Iran (cf. even Wright 1975).

12 The number 1ooo is based on statistical inference (Vaiman 1974, 17). An earlier and less rigorous estimate goes as far as 2ooo signs (Falkenstein 1936, 22-28). In any case, the number of basic signs (which could be combined) must exceed the 46o already testified (Vaiman 1974, 17).

Even the number of material tokens of the precursor of real writing is impressing. Denise Schmandt-Besserat (1979, 23) had studied c. 2oo different types.

13 Cf. Falkenstein (1936, 43-47; 1953, 134) and Edzard (in Cassin 1965, 46). The evidence consists of tablets containing systematic word lists, and it is eo ipso dependent on writing on tablets; whether even the "token-script" and its use was learned in the same way is therefore an entirely open question.

14 This common development is valid only for Sumer and Sumerian dependencies in Northern Mesopotamia and Syria; though clearly related to Sumerian script and notation in the beginning, writing in Elam in Southwestern Iran took different directions, thereby pointing to cultural separation (in the case of metrological notations, this is clearly shown by Friberg (1978)).

15 Cf. Falkenstein (1936, 45f).

16 Caveat: This statement is only supported by indirect evidence. It is very difficult to prove that the coordination revealed in the early written sources was not in existence in pre-literate times. Part of the development may have taken place already in the era of the token-notation; indeed, it seems plausible that the above-mentioned use of capacity measures in simple ratios and the laying-down of temple plans by geometrical constructions constitute the earliest steps on the path toward integration of mathematical techniques into one system.

17 Cf. Vaiman (1974,19).

18 Cf. Falkenstein (1936, 49f) and Friberg (1978, passim). A somewhat but not very much later development of the weight system along similar lines is conjectured by Powell (1971, 2o8-211).

19 Powell (1972a) discusses this feature of Sumerian area measures extensively (see pp 174, 177, 218f, and passim). Vaiman (1974, 19f) mentions a number of tablets from Jemdet-Nasr (c. 3ooo B.C.) where the calculation of areas from linear extensions is clearly used. Work in progress (Jōran Friberg, Denise Schmandt-Besserat) may or may not carry this mathematically based area system back to the late fourth millenium. It may perhaps even change some details of Powell's argument, - but probably not what is the main point in this connection.

The use of area measures based on measures for length may seem trivial to those of us who are fostered with the metric system. Still, we should remember that the English acre (defined as 22o yards long by 4o broad) is much closer to the "natural unit" on which it was based ("as much as a yoke of oxen can plough in a day" - SOED, article "acre"). Than the Sumerian "garden plot" (šar) which as early as we know it had come to mean 1 square nindan (cf. Powell 1971, 19o, 219).

2o More detailed expositions of the development of Sumerian society in the third millenium are found in many survey works. I shall only refer to Cassin (1965), Garelli (1969), Kramer (1963), Diakonoff (1969, 7o-87 and 173-2o3, articles by Tyumenev and Diakonoff) and CAH I^2.

21 Most of the essential sources were collected by Thureau-Dangin in his collection of royal inscriptions (19o7). Supplementary texts are given by Sollberger (1971). For an important text dealing with an early real or postulated social reform, see M. Lambert (1956), Diakonoff (1958) and Brentjes (1968, 259); cf. also Tyumenev (1969a) who deals with the real contents of the reform.

Almost the same ideology is expressed in the bulk of Sumerian literary texts, many of which deal with the advantages of civilized life (irreducibly dependent on the organization through the state). This is an important theme in the proverb collections, as the one published and discussed by Alster (1974, 1975); it is also to be found in lamentations over destroyed cities (one instance in Pritchard 195o, 455-463) in myths dealing with the origin of civilization (several are published and discussed in Witzel 1932), and in lots of other didactical and religious texts.

22 See the inscriptions translated in Sollberger (1971, 97-1o8). The close connection between Sargonic empire-building and foreign trade is emphasized by a tradition according to which Sargon seems to have invented the "protection of American/Akkadian life and property" as a valid argument for war and conquest (C.J. Gadd in CAH I^2, 426f).

23 On the rise of the organized profession of scribes around 25oo B.C., see Tyumenev (1969, 77) and M. Lambert (1953, I,198; II,151f).

24 What is said concerning the role of the education of scribes is still a reconstruction, built on lexical lists, changes in the script, the social organization as revealed in economic texts, etc. Cf. Falkenstein (1937, 46f) and M. Lambert 1953, II,151f).

25 Cf. Benno Landsberger in Kraeling (196o, 11of). On the illiteracy of priests, judges, governors and most of the population in general, see ibid (98f) and Kraus (1973, 19f).

Concerning the reality behind the few royal claims of literacy, it seems that Ašurbanipal (7th century B.C.) was able to read simple texts; the context in which Sulgi of Ur III (c. 2o5o B.C.) made his claim makes its veracity most doubtful; the same applies to Lipit-Ištar's assertions (193o B.C.).

26 Cf. Oppenheim (1965) and Landsberger in Kraeling (196o, 97).

27 It is noteworthy that exactly in Ur III the scribe "could climb to the highest administrative posts" (ibid, 99) instead of remaining at the subordinate level. No other Mesopotamian state formation kept so closely to the managerial ideology as did the hyperbureaucratic Ur III.

28 According to hymns made in the name of king Šulgi, the curriculum of the Ur III school contained writing, arithmetic, accounting, field-measuring, agri-culture, construction, and a few subjects the names of which are not understood (cf. Sjōberg 1976, 173f). Other later texts (see Sjōberg 1975, 145, Kramer 1949, 2o6, and Falkenstein 1953, 126) confirm the importance of mathematical subjects. As far as the school before Ur III is concerned, only the occurrence of school tablets containing mathematical exercises/cf. Powell 196, passim) but no surveys of curricula exist which might confirm the importance of mathematics.

29 Cylinder A 19, 2o-21; translation Thureau-Dangin (19o7, 111). A sort of theological numerology, ascribing to each god a sacred number, was current in Mesopotamia, probably from the mid-third millenium (cf. Ur-Nanše, stone tablet A, III, 3, in Sollberger

1956, 3, and annotated translation in Sollberger 1971, 45f) until late (cf. Labat 1965, 257f). Since this type of sacred mathematics was intermingled with the calendar (cf. Landsberger 1915, e.g. pp 114, 127, 13o and 137), even theological numerology was a scribal concern, as also suggested by Gudea's text.

3o E.g. the occurrence of mathematical school exercises in Šuruppak around 25oo B.C. (cf. Powell 1976, 429-434), exactly in the context where it is for the first time possible to establish the existence of a professional body of scribes (cf. note 23).

31 "It is in the Šuruppak records and documents that the organization of the temple estate is for the first time exhibited in its basic features with sufficient distinctness and clarity" (Tyumenev 1969, 76). Besides this more systematic administration of established social management, Šuruppak is also the first place where a greater part of the working population is provided for by distribution of rations in kind - a system which required a considerable amount of administrative order, and which until Ur III was to become steadily more important (cf. Gelb 1965). In this connection it is not without interest that two of the mathematical exercises from Šuruppak discussed by Powell (1976, 432-434) deal with the distribution of rations of barley.

On the probable connection between the somewhat later rise of a genuine royal city-state and further advances in scribal culture (in this case, improved writing), see M. Lambert (1952, 75 f).

32 Cf. note 3o and 31. Further evidence can be extracted from Powell's treatment of Sumerian weight and area metrologies (1971; 1972a).

33 "auch die kleinen und die kleinsten Transaktionen wurden schriftlich dokumentiert, was eine so detaillierte Buchführung zur Voraussetzung hatte, daß wir sie sogar heute als überspitzt bezeichnen müssen" (N. Schneider 194o, 4).

34 Cf. for instance Landsberger (in Kraeling 196o, 111). Šulgi was one of the first Near Eastern despots to proclaim his own divinity (cf. Kramer 1963, 62-69), a symptom perhaps of the new status of the royal state.

35 Falkenstein 1953, 128. On the reality behind the claim, cf. note 25.

36 Cf. material presented by Powell (1971, 1972a, 1976).

37 For a discussion of earlier theories of this transition and a rounding-off of the debate, see Powell (1972, 1976, 418-422). Ellis (197o) has published a tablet (written no later than 6 years after Šulgi's death), where place value numbers were used for marginal calculations, while the main text of the tablet writes its number in the hitherto current notation.

38 The earliest dated use of sexagesimals (the one mentioned by Ellis) has 1,3o for $1\frac{1}{2}$. Neugebauer (1935, 1o-12) lists a number of tables of reciprocals (division by n was replaced by the multiplication by $\frac{1}{n}$ in the sexagesimal place value system) which to judge from the script may belong to the early lifetime of the place value system.

The early occurrence of tables of reciprocals suggests (if it is really as early as it seems to be) that the immediate introduction of sexagesimal fractions was perhaps not just a by-product of the lack of a "sexagesimal point". It is possible that it is the realisation of the possibilities of reciprocals which sparked off the creation of the place value system. It is indeed probable that the Sumerians had since long noticed that divisions by 2, 3, 4, 5, 6 and 1o were almost equivalent to multiplications by 3o, 2o, 15, 12, 1o and 6 in the earlier number notation. (Landsberger 1915, 13o, suggests the use of this sort of multiplicative complementarity in the theological numerology). On the other hand, the generalisation of this principle was probably only possible by means of a new adequate representation of numbers - i.e. the place value system.

39 One system is that of the balanced account, which for a stock of goods or a capital of money (i.e. silver) states the capital (previous balance plus incoming quantities), expenditures and their sum, and the new balance (see Fish 1938, 166-17o; id. 1939, 32-37); another system, involving cross-wise control of the overseers who managed the royal estates, was described by Struve (1969, 147f).

4o Needham 1954, III, 8-13, 36, 45f, 83-89. The process is well on its way in the 14th century B.C. and is only formally finished in the 14th century A.D.

41 This is at least the opinion expressed in Closs 1977, 25f. Still, it is possible that the Mayas' occasional break with the pure base 2o-system was due to practical reasons belonging to their astronomy, and not to any lack of understanding - cf. J. Lambert 198o.

42 Cf. R. de Roover (1937, 1956). It seems that double entry book-keeping was developed rather spontanously in several Italian cities around 13oo and spread slowly through Italy (1937, 115-159) and to the connected commercial centres in Flanders (where the system was used as early as 137o - ibid, 164f) and Southern Germany (where primitive versions of the system were in use in the mid-fifteenth century - ibid, 171f). But even in Milan and Florence, the system was only gaining general foothold during the fifteenth century (ibid, 154f), and as late as the sixteenth century Hanseatic commercial expansion was fettered by inadequate bookkeeping (ibid, 165-17o), which, like Šuruppak or pre-Šuruppak Sumerian accounting, was in principle nothing more than a set of memoranda to support memory (cf. also ibid, 182-185). And, in cases where full or partial double-entry bookkeeping was really used, it was often done in ways which missed the essential points (no summing, no checking, no balancing), cf. Roover (1956, 162f) and Ramsey (1956, 196f). As late as the 16th century, most Western European merchants' books

were "so grosly, obscurely and lewdely kept, that after their desease nether wife, seruaunt, executor nor other, could by their bokes perceive what of right apperteigned to them" (Jean Ympyn, A notable and very excellente woorke, expressing and declarying the maner and forme how to keep a boke of accomptes ..., 1547; quoted from Ramsey 1956, 186). Which enormous difference from the precise and systematic Ur III book-keeping.

43 It is of course not impossible that e.g. the invention of the sexagesimal place value system was made by some practitioning scribe of genius; but had not the school been there to teach everybody the devices of place values and sexagesimal fractions and to provide them with the neccassary tables, he would have been bound to use his inventions only in his private intermediate calculations; any written communication with other scribes would have to employ established notations; diffusion would have been slow if not outright impossible. Further, since the system was only advantageous when supported by extensive tables, the individual inventor would have had no reason to use his own invention, had he not had the backing of the school where these tables could be made (and their use taught). So, it seems most plausible that even the invention itself was made inside the school system. In any case, since the first occurrences of sexagesimals were in marginal and intermediate calculations the results of which were then afterwards translated into established notation (Ellis 197o, 267; Powell 1972, 14f; Powell 1976, 435 n. 6), it seems to be almost sure that the system was taught to future scribes as a professional aid.

44 Kraus 1973, 23-25. There are textual reasons to believe that the official names for years, royal inscriptions and royal hymns already in pre-Ur III times came "aus derselben Werkstatt" (ibid, 124; cf. Hallo 1976, 185 and passim), and that this workshop was not foreign to the tablet house in Ur III (Sjöberg 1976, 171, 174). It seems inherently plausible that even the book-keeping systems were not only taught but even created in the tablet-house on "public" (i.e. court) demand. At least it seems obvious from a comparison with the Mediaeval diffusion of the double entry system that a social need for accounting has little effect if it is not supported effectively by an institution which is able to diffuse the techniques in question. Indeed, the only place where the diffusion of double entry accounting was not extremely slow was in Italy, where it was taught in the "abacus-School" (Roover 1937, 282, 29of; Fanfani 1951, 337-342; Thorndike 194o, 4o2; M.D. Davis 1977, 11-18; Goldthwaite 1972), which, though not strongly institutionalized, was a school; when, finally, the breakthrough took place in the late Renaissance, it was heavily supported by the art of printing which made possible new forms of instruction (Roover 1937, 29o and passim; Eisenstein 1979: I, 382f).

45 True, the material is full of quite random lacunae; in principle, "pure mathematics" of the sort later known from Babylonia may suddenly turn up in the older Sumerian strata. But comparison with the distribution of literary texts seems to indicate that Sumerian precursors of later Babylonian pure mathematics are really non-existent: Indeed, even if most literary texts are known only from later versions although they have Sumerian or Sargonic origins, a reasonable number of literary tablets are known illustrating the tradition all the way back to 25oo B.C. (Hallo 1976, passim; Alster 1974, 7). No such precursors for Babylonian pure mathematics exist.

46 See Powell 1976, 428f.

47 See Powell 1976, 432f.

48 Cf. note 28 for the school curriculum as depicted in two of Šulgi's royal hymns. It can be noticed that whatever literary tasks fell upon the tablet house were according to these texts not included in the curriculum but a task incumbent on the staff (cf. also Sjöberg 1976, 171f, where two other hymns are quoted to this effect).

49 This integration can be deduced from the fact that most remains of scribal activities known from Sumer and almost all from Ur III are connected with the activities of "public" institutions (temples, royal domains, state trade, state industries) or, in late Ur III, with enterprises which were formally public property even if in reality the private possession of the managers (cf. Garelli 1969, 1oo-1o4; Leemans 195o, 42-48).

5o Not just by the weight of the bureaucracy, but because of the much too great strain to which the population and probably even the land were submitted by the whole exploiting machinery of the Ur III-state - cf. Diakonoff (1971, 2of); Lamberg-Karlovsky (1976, 67f).

51 The change in direction of a private money economy was general: The semi-enslaved workers receiving rations in kind were replaced by free labourers working for a wage (see Gelb 1965, 23o, 242f); silver came into more general use as standard of value, i.e. as money (Klengel 1974, 252); private possession of large-scale landed property became common, public foreign trade was replaced by private trade, and a sort of banking was developed (cf. Leemans 195o, 62f, 113-125; Dandamajev 1971; Edzard in Cassin 1965, 196-199). Still, we should not over-emphasize the modern and capitalistic aspects of the Old Babylonian economy; trade in land, which was the basic means of production, was perhaps free but not on market terms (cf. Jakobson 1971), and each time the Old Babylonian state grew sufficiently strong, it tended to control the private sector (cf. Dandamajev 1971, 69; Leemans 195o, 114-117). For the general trend, cf. even Garelli (1969, 11of) and Klengel (1974).

52 Cf. Klengel (1974, 252-254; 1977) and Kraus (1954, 537; 1973, esp. 13o-143, but even 4o-42 and the work in general).

53 This might seem a paradoxical statement, given the cultural continuity of Mesopotamian society. It is not - just consider the case of the renaissance, this era of supreme free creativity disguised as a return to the past. More precisely to the point, see Kraus' discussion of the Old Babylonian use of older mythology (1973, 13o-134) and the fact that the Old Babylonian period witnessed the creation of a quite new epic literature in the Akkadian language (see Hallo 1976, 199).

54 There is an interesting parallel between the street scribe, writing letters and necessary documents for the common man and woman for money (Landsberger in Kraeling 196o, 99) and the exorcising priest and diviner leaving the temple to work for private clients (Kraus 1954, 537). Both make their appearance in the Old Babylonian period.

The scribes' detachment from the public functions is, however, not restricted to the case of the notarial activity and private letter writing. Indeed, the accounting techniques invented for public purposes in Ur III were amply used even in private business in Old Babylonia (cf. Edzard in Cassin 1965, 194, and material presented in Leemans 195o, passim).

55 The evidence, however, is anything but clear - cf. Sjöberg (1976, 176-178) and Falkenstein (1953, 125f).

56 The precise chronology of the process is at least for the moment not to be known. Still, the decisive steps must be placed in the earlier part of the period, since a number of characteristic texts dating c. 18oo B.C. have been found in Tell Harmal (See Baqir 195o; 195oa; and 1951; Goetze 1951; and al-Hashimi 1972). This agrees with Neugebauer's paleographic estimate for the date of an early characteristic text (1935, 117).

A limit post quem is obtained from the observation that the problem type most characteristic of the new pure mathematics - the "equation" of the second degree - is intertwined with the use of the full potentiality of the sexagesimal number system; it seems (as is also indicated by the lack of such texts in strata before the Old Babylonian period, cf. note 45) that this type of mathematics can only have been developed after the use of sexagesimals had been generalized.

For linguistic reasons, the geographical origin of the new mathematical development of the early Old Babylonian period must be placed in the South, e.g. around Larsa (see Goetze in Neugebauer 1945, 146-151). The language was Akkadian, the language of the new literary creativity.

57 Instances of seemingly pure geometry are published by Saggs (196o - see also Caratini 1957 and Neugebauer 1935, 137-142 on the same text) and Bruins & Rutten (1961, 22-24 and plates 1-3). The texts concern regular polygons and patterns made from squares and circles.

These calculations of the properties of beautiful patterns are clearly peripheral to the main concerns of Babylonian mathematics. In contradistinction to the algebraic problems (cf. below) they may also be a direct continuation of a much older tradition - one tablet from Šuruppak (Jestin 1937, no. 77) may be a precursor.

58 This assertion only holds true under one condition: A specific division between geometrical and non-geometrical mathematics which may seem artificial but which I make on the basis of my impressions of the structure of Old Babylonian mathematics as a whole.

As geometry I define problems where the interest in visual form seems to define the problem. For such problems, see note 57.

As algebra I define all problems of the type "To my square I add 1/2 of my side: 2o". The addition of area and length clearly shows that what seems geometry is just a way to express powers and products. Extrapolating this, a lot of similar but homogenous problems are also considered algebraic.

In the borderland between the two we find a large number of problem types dealing with plane or spatial structures: Division of triangular "fields", calculation of the sides of trapezoids and the volume of various constructions. Because of the treatment given to them and the emphasis of the interest, I will consider them as applied algebra.

Finally, a certain number of problems deal with proportionality between equiangular triangles. They may approach the above definition of geometry, as being defined by visual form (for one early instance, see Baqir 195o, and discussion by Drenckhahn 1951). On the other hand, the technique of proportionality as exhibited here is inside the entire pre-modern mathematical tradition (apart the important Greek interlude) a legitimate and important part of algebra. Problems of this type may even have been among the starting-points for the development of the Old Babylonian algebraic techniques. Therefore I tend to see even them as belonging to algebra. If instead we regard them as geometry, we should come to a result quite different from the methodical insignificance of pure geometry: In this case, pure geometry was probably a most important starting-point for the development of algebra.

59 This dominance of algebra is born out by a survey of any of the greater collections of Babylonian mathematics, i.e. Neugebauer (1935 + 1935a + 1937), Thureau-Danging (1938), Neugebauer & Sachs (1945), Bruins (1961), and even by the "mathematical compendium" from Tell Harmal (Goetze 1951). The problems of the second degree are all solved by appropriate general methods.
Problems of the third degree are sometimes solved by reduction at a standard form

($x^3 + x^2 = a$, expressed anachronistically)

and use of a tabulation of $n^3 + n^2$, and the fourth, sixth and eighth degree by treatment as second-degree problems in powers of the unknown. In general, the particular solutions to specific problems of a higher degree demonstrate that the Babylonians were unable to cope with problems exceeding the second degree - as remarked by Thureau-Dangin (1938, xxxviii) for the third degree.

General treatments of Babylonian algebra are found by Neugebauer (1934, 175-2o2), Vogel (1959, 45-64) and Goetsch (1968).

6o VAT 8528 no. 1 and AO 677o no. 2, e.g. in Thureau-Dangin (1938, 118-12o and 72).

61 This applies to the bulk of second-degree problems. Where, for instance, would a practitioner meet four squares, of which he knew the total area to be 1626 and the ratio between the sides to be 6o, 4o, 3o, 2o, without knowing the dimensions of the single quares? (BM 139o1, problem no. 15, e.g. in Thureau-Dangin 1938, 7).

The practical character of the second-degree problems might at a pinch be defended by the argument that they were uses to train general algebraic techniques (even though the one-eyed concentration on useless second-degree problems undermines the argument). But the interest in problems of a higher degree, of which single specimens are solved by methods which only apply in these specific cases, has no excuse of that sort. The dimensions of a rectangle of which the area and the product of the cube of the length and the diagonal are known (Susa-text XIX, problem D, in Bruins 1961, 1o3-1o5) are only interesting in one respect: That they can be found, in spite of the seemingly complicated character of the problems in general. From the aesthetic point of view, the term virtuosity comes to the mind rather than beauty.

62 In fact, the interest in exactly those problems whose solution is possible by means of the methods which one has at hand is specific for Old Babylonian mathematics. Pure mathematics of the Greek brand (and its descendants) is (at least in principle) characterized by the development of new methods which solve problems which are interesting as problems - either problems existing before theory is devloped, like the Delian problem (doubling of the cubic altar in Delos) or generated by mathematical theory (Archimedes' calculation of the area of the parabola).

63 Published for instance in Pritchard 195o, 163-18o. The continuity in the concept of the public authority is also confirmed by the attempts of all Old Babylonian rulers strong enough to envisage such steps to control the private sector of the economy (cf. above, n. 1o.3). A royal edict concerning the annihilation of debts from c. 163o B.C. demonstrates a continuity in royal phraseology extending to the very and of the Old Babylonian period (see Edzard 1974, 151-153).

64 Many of these sources are texts which were seemingly used in the scribal school to inoculate that same professional pride. Sjöberg has published a number of such texts (1972; 1973; 1975), another composition ("Schooldays"), going back perhaps to the end of Ur III and frequently copied in later times, was published by Kramer (1949). For discussion, cf. Falkenstein (1953, 133), Landsberger (in Kraeling 196o, 94-1o2, and subsequent discussion) and Lucas (1979).

65 See the material mentioned in the previous note. Further material, pointing to the importance for professional ideology of the secretarial functions in political management, is presented by Oppenheim (1965).

66 Just as it had done in king Šulgi's days - cf. note 28.

67 So, king Ašurbanipal (cf. note 25) claims in his boasts of scribal cunning, that "ich löse komplizierteste Multiplikations- und Divisionsaufgaben, die sich nicht durchschauen lassen" (Falkenstein 1953, 126). Admittedly, this is a late text, but one reflecting that earlier scribal culture which the king tried to preserve in his library. In "Schooldays", a father, praising the teacher for learning his son the best of the scribal art, explains the detailed meaning of this by specifying that "der Rechentafeln, des Rechnens und Abrechnens Lösungen erklärst du ihm, der Divisionen (?) verschleierte Fragen läßt du ihm aufgehen" (translation by Falkenstein, 1953, 129). Cf. also note 68.

68 "Examination Text A", published by Sjöberg (1975). See also Landsberger in Kraeling 196o, 99-1o1.

69 "Mathematics" is asked for in the following formulation: "Kennst du die Multiplikation, die Bildung von reziproken Werten und Koeffizienten (a technical term), die Buchführung, die Verwaltungsabrechnung, die verschiedensten Geldtransaktionen, (kannst du) Anteile zuweisen, Feldanteile abgrenzen?" (Translation Sjöberg 1975, 145).

A much shorter but spiritually related list "(Examination Text D", "In Praise of the Scribal Art", published by Sjöberg, 1972) culminates with official stele writing (for the king), surveying, accounting and some sort of official service (p. 127).

7o See note 28 and 64.

71 Most obviously, the occult languages mentioned in Examination Text A. But even the mastery of the Sumerian language, which in Old Babylonian times was a dead literary and administrative language, should be mentioned as a thread in the pattern, not least because the mastery of Sumerian played an important role for professional self-consciousness - the bad scribe was "no Sumerian" (see e.g. Landsberger in Kraeling 196o, 96f). More closely to the point is, however, the adaptation and canonization of Sumerian literature, and the training in "composition of poetry in a highly artificial Sumerian" (ibid, 97; see also Hallo 1976, passim). Landsberger estimates (in Kraeling 196o, 11o) that all this literature was not intended for wider circles, but

was pure l'art pour l'art, which the scholars might use "to retain their own importance as a closed corporation". In the same vein, Speiser mentions (ibid, 107) as an example a scribe who, inscribing a royal statue, "insisted on showing off what he knew. Following the principle of not being simple when he could be complicated, he scarcely ever wrote the same sign twice in the same form".

72 On this concept of scribal "humanism", see Sjöberg 1973, 125, comment to line no. 70 (the reference to line 72 should read 71). It seems to be specific for the Old Babylonian scribe - at least it leaves no traces in the Ur III royal hymns concerned with the edubba. Maybe it was only inside the mental framework of Old Babylonian individualism that the scribes would get the idea of being humans par excellence.

73 Of course, the virtuoso's tricks are also present, namely on the front of higher-degree problems where no general breakthrough took place - cf. note 61.

74 BM 85194, no. 25, e.g. in Neugebauer 1935, 149, 160f, 182-184.

75 The predominance of technical terminology in Old Babylonian mathematics can be evaluated by a glance at the transcriptions in Neugebauer's Mathematische Keilschrift-Texte (1935 + 1935a + 1937). Only the words written in italics are in the basic language (Akkadian). The numbers correspond to the Sumerian number system (Akkadian spoken numbers were decadic), and Roman and capital letters stand for Sumerian words and for ideograms. The Akkadian syllables constitute perhaps one third of the whole text; the rest (and even part of the Akkadian words) must be considered standardized vocabulary.

The extent to which this standardized vocabulary was abbreviated and stereotypized is illustrated by an analysis made by Neugebauer (1935b), where "šá uš-šè íl" is demonstrated to stand for "igi-TE-EN šá sag uš-šè íl"(pp. 243-246).

The age of at least part of the technical vocabulary as a standardized terminology is demonstrated by its immunity against a change in the direction of the script which took place presumabl shortly after the Šuruppak period (Falkenstein 1936, 11; the immunity of terms like "upper breadth" and "height" is noticed in Thureau-Dangin 1938,xvii and Vogel 1959, 15).

76 For this harsh picture, see for instance Cassin (in Cassin 1966, 50-54).

77 Cf. ibid, 65-67, and Landsberger (in Kraeling 1960, 97). The theme of the sufferings of the just had been treated already in the Old Babylonian literature, but as a theme for protest and revolt (a Syrian reflection of this genre are Rib-Adda's letters, see Liverani 1974); now the theme changed into one of resignation and fatality - see Klengel (1977, 115). When justice could no longer be legitimately claimed from the state, how should it then be possible to raise such a claim on the gods?

78 Cassin (in Cassin 1966, 53f).

79 Ibid, 65. It seems that the Kassite kings kindled this feeling, cf. Hallo (1976, 201).

80 See W.G. Lambert (1975, 3-6). Lambert identifies scribal families active around 200 B.C. as descending from early Kassite "ancestors".

81 For the fusion of scribal and priestly functions, see ibid, 4-5. For the disappearance of occupational specializations, see Landsberger in Kraeling (1960, 97).

82 In 'Mathematische Keilschrift-Texte' Neugebauer (1935 + 1935a) locates a number of problem texts as "etwas Kassitisch oder etwas älter", judged from the writing. All of these, Thureau-Dangin (1938, ix) locates in the Old Babylonian period. Goetze (in Neugebauer 1945, 151) judges them to be northern and slightly younger modernizations of South Mesopotamian originals from the early Old Babylonian era (which should make them late Old Babylonian). One text both Neugebauer and Thureau-Dangin judge to be probably Kassite (AO 17264). One series of texts Neugebauer (1935, 383-516) situates for lack of other convincing dating in the Kassite period, while Goetze leaves them out from his linguistic analysis, presumably because their language admits no chronological or geographical localization. Later, Neugebauer seems to have revised his opinion on most of these dating them back in the Old Babylonian eara (1969, 29).

The disappearance of problem texts from the sources might of course be due to particularly bad luck. In general, sources are relatively scarce from Kassite times. But Ašurbanipals boasting of quite elementary mathematical training (cf. note 67) constitutes independent evidence that highlevel mathematics had disappeared from the mental horizon. In fact, the context shows the king boasting of the most improbable abilities; had he known of algebra he would certainly have asserted mastery of even this field.

83 A rather large number of tables of reciprocals and especially multiplications tables are dated to Kassite times by Neugebauer (1935, 11, 36-42), although with serious reservations (see also Neugebauer 1945,1 note 4). Kilmer (1960) has published a Kassite list of technical coefficients of a type also current in Old Babylonian mathematics (diameter of a circle with periphery 1, bricks carried per man per day, etc. - cf. Neugebauer 1945, 132-139), but interesting especially because it seems to mix mathematical coefficients with material of a non-mathematical character. If this is so, and if no comparable mixed texts existed in the Old Babylonian period, this might indicate a breakdown of the strict structure of Old Babylonian mathematics.

84 The same seems to be demonstrated by the elements of Babylonian scribal culture taken over by various Syrian scribal schools. Apart the Akkadian language (which was a common means of diplomatic expression) the sources show education in letter-writing, Babylonian "wisdom-literature" and theology - but the only mathematical

texts found are "Einspaltige Tabellen von Gerstemengen (Hohlmaß), Silberbeträgen (Gewicht) und Felder-Grundstücken (Flächenmaß)" - see Krecher (1969, 149).

However, the states in question were centralized palace economies, with an important role to play for the scribal managers. Evidently, these needed advanced mathematics just as little as did the Kassite scribes.

85 For one thing, the school (i.e., in this period, the scribal family) seems to have had pretty little to do with this revival, which was rather due to the rise of mathematical astronomy (and so, ultimately, to the existence of officially patronized astrology and calendar science). Besides, there were few new theoretical developments in late Babylonian mathematics. One innovation is the technique of interpolation extensively used in lunar and planetary astronomy (see Neugebauer 1969, 11o-118). It is clearly a product of the astronomical application, not of institutionalized teaching. Another possible innovation is the summation of complicated progressions:

$$\sum_{i=1}^{10} 2^{i-1} \quad \text{and} \quad \sum_{i=1}^{10} i^2$$

(Neugebauer 1935, 96f, 99, 1o2). At least, these are only found in a late Babylonian problem text, and even the summation in both cases of exactly 1o members (which is also current in later Islamic summation of the same progressions) may perhaps indicate a late date. If this is an innovation, it could be a "pure" spinn-off from the application of arithmetical progressions in mathematical astronomy, and so it might have to do with mathematics teaching for astronomers - but this is pure, unsupported guess and hardly worthwhile pursuing. It should just be mentioned that Goetsch (1968, 98) points to a possible (but hardly convincing) connection between the summation of i^2 and a couple of Old Babylonian problem texts (YBC 7417 no. 1-3, in Neugebauer 1935, 498, and BM 139o1 no. 18, in Neugebauer 1937, 4, 9); so, innovation at this point in late Babylonian times is by no means assured.

86 The Egyptian influence is less perceptible than the influence from Mesopotamia. Anyway, the existence of some continuous traditions from Middle Kingdom Egypt via Greek, Islamic and Hebrew mathematics to Medieval Europe can be demonstrated beyond reasonable doubt. Apart the use of unit fractions (on whose continuity see e.g. Tropfke 198o, 94-113, passim, or any other survey of notations in ancient and medieval mathematics) and certain techniques of handling them (the "reference number" or "bloc extractif", on whose continuity see Rodet 1881, 196-232), a specific way to formulate certain algebraic problems of the first degree and to solve them by "single false position" (cf. Rodet 1881, 4o1-447, and the important Papyrus Akhmîm, published by Baillet 1892). Even the "finger-reckoning" known from Antiquity and from Islamic and Western medieval mathematics seems to descend from Egyptian practices (see Menninger 1957, II, 3-24, and especially the Egyptian cubit rod on p. 24 which seems to suggest a relationship between Egyptian finger symbolism and later practices.

87 The most striking indication of Egyptian independence is the Egyptian number system. It is purely decadic, with signs for 1, 1o, 1oo, 1ooo, 1o ooo, 1oo ooo and 1 ooo ooo, already found in the earliest document known which contains numerals (Sethe 1916, 2), from c. 31oo B.C. (Hayes in CAH I[1], 174) (On the earliest Sumerian and Proto-Sumerian number systems, cf. Friberg 1978; they are totally different). Another indication comes from the fundamental difference between Egyptian and Mesopotamian arithmetical techniques (Reineke in LdÄ III, 1238, article "Mathematik"), however, since the arithmetical techniques are in both areas only known from around 2ooo B.C. or later, this sort of evidence is not very heavy.

A few connections may exist; so, a dyadic system of sub-units used in Egypt for corn and for areas (cf. Gardiner 1957, 197, 266) has a parallel in proto-Sumerian mathematics (cf. Falkenstein 1936, 49). This may, however, just as well be a mere coincidence, and in any case it is rather insignificant. In other cultural areas certain connections are attested (cf. Vercoutter in Cassin 1965, 231; Edwards in CAH I[2], 41-45).

88 See Iversen 1975 - on the age of the system, especially pp 6o-65. For statues, the grid of squares is used in connection with a technique of orthogonal projection, which is also the basic principle in Egyptian architectural design (see Badawy 1948, 264-265).

89 See Reineke (1978, 7o-71). No direct evidence exists concerning the late fourth millenium metrology, only plausible reasoning founded on general knowledge of Egyptian society in that period (which is quite restricted) and on later metrology.

9o See Reineke (1978, 73-75).

91 Just see the Old Kingdom statues of self-assured scribes and other officials, e.g. Breasted (1936, Abbildungen 63-67 & 69-7o).

92 See Wilson in Kraeling 196o, 1o2f.

93 So, Reineke 1978, 76, who points to the parallel development in art and material culture.

94 An impression derived in part from the sources (Old Kingdom texts like the Abu Sir papyri, see Posener-Krieger 1968, 1972 and 1976, and even Silberman 1975, compared with later administrative texts like the Reisner Papyri, see Gillings 1972, 218ff, which use a much more elaborate mathematical apparatus); in part from Sethe's classical treatment of the Egyptian number system (1916); and in part from the organization of the problems of an applied character in the Rhind mathematical Papyrus (Peet 1923; Chace 1927, 1929). It is for instance worth noticing that of 6 problems concerned with the (inverse) slope of pyramids those five which may seem to be of Old Kingdom origin (since the slope is close to that of real Old Kingdom pyramids, cf. Reineke 1978, 75 n.28) give the result in a practitioner's way: e.g. 5 palms 1 finger per cubit height. The sixth result, evidently not related to the old pyramids, is given in pure number.

95 The last trace of the tradition is the Akhmîm-papyrus, written in Greek - see Baillet 1892. Intermediate steps in the development are found in the Demotic mathematical papyri (Parker 1972).

96 These are my private impressions from comparisons between the later texts (cf. note 95) and the Rhind Mathematical Papyrus, as supported by the conception of Middle Kingdom mathematics as a coherent structure. I should point out that what I consider a new element in Demotic mathematics (i.e. factorization) is believed by Kurt Vogel (1974) to be present behind the curtain already in the Middle Kingdom.

The concept of coherence, as applied to Middle Kingdom mathematics, has two sides. First, any reasonable calculation performed inside the framework of this mathematical structure leads forward to an answer which can itself be formulated with exactitude inside its limits; this is only true when the full Middle Kingdom system of unit fractions (see below) is used (as is the case even in administrative Middle Kingdom texts), not with the more restricted mathematical habits of the Old Kingdom texts; here only metrological subunits and simple unit fractions (and few sums of unit fractions) are found. Second, the way in which the unit fraction system is used until the very end of the Egyptian mathematical tradition is in harmony with the additive and scaling structure which dominates Middle Kingdom mathematics; in the context of the multiplicative understanding revealed in later texts it becomes unnecessarily clumsy, and it is indeed supplied with (but not replaced by) a notation approaching that of common fractions (see Parker 1972, 8f; for a particularly striking case of conceptual hybridity, see p. 66f, problem no. 57).

In certain respects, the coherence of Middle Kingdom mathematics is virtual rather than actual. This seems to demonstrate the chronologically secondary character of the coherence. Indeed, a number of e.g. Rhind Papyrus problems are solved inside the coherent framework of pure numbers and afterwards translated into the language of current and probably older metrological units (two instances of a quite different kind are no. 37 and no. 41). Another violation of coherence (also found in no. 37) is the use of the expression "1/3 of 1/3" in the formulation of a problem; I would guess that it stems from a habit of ordinary language - in any case, this way to express numbers is current in the related Arabic language (see e.g. Saidan 1974, 368). So, both breaches of coherence I would ascribe to one type phenomenon - viz., the attempt of the text to integrate practical mathematical problems and daily-life mathematical idiom into the structure of theoretical mathematics.

97 Rhind Mathematical Papyrus, problem no. 33.

98 Wilson (in Kraeling 196o, 1o3).

99 See e.g. the "Satire on the Trades" (Pritchard 195o, 432-434). The text probably originated in the Middle Kingdom.

100 A particularly high-flown text compares the fame obtained from scribal scholarship with the permanency of tombs and pyramids - to the advantage of scholarship (and in fact we still know several of the scholars mentioned by name in the text) (see Pritchard 1950:431-432). This text too was used in school to impregnate professional pride.

101 In the "Satirical Letter" (Gardiner 1911; even this text was used in school for purposes of professional indoctrination), lack of applied mathematical ability is emphasized as a main characteristics of the incapable scribe. The extent to which mathematics was (over-)appreciated by those knowing it is perhaps best indicated by the introductory words from the Rhind Papyrus: "Accurate reckoning of entering into things, knowledge of existing things all, mysteries ... secrets all" (translation Chace 1929).

102 The text in question is a Middle Kingdom temple account (see Gillings 1972:124ff). Apart the absurd combination of meticulous precision and gross unnoticed errors the practical superfluidity of the theoretical refinements of the unit fraction system is also seen in the very rough approximations which turn up in certain cases - so in a New Kingdom text (late second millenium B.C.) using an exchange rate between copper and silver of 10:1 (measured in different units), but none the less equating 18 units of copper with 1 2/3 units of silver instead of 1 4/5 (1 2/3 1/10 1/30), and both 14 and 16 units of copper with 1 1/2 unit of silver, instead of 1 2/5 and 1 3/5 (respectively 1 1/3 1/15 and 1 1/3 1/5 1/15; all unit fraction expressions correspond to the canon of the Rhind Mathematical Papyrus). See Frandsen (1979:283).

103 Wilson (in Kraeling 1960:103).

104 By the term "Greek mathematics" I shall refer to mathematics written in Greek during the whole period of Classical Antiquity, be it in Greece or in the Hellenistic or Roman world. As far as they are worth mentioning, even works in Latin are included.

105 Not only is this point of view expressed by Plato, in whose philosophy the distinction between the perceptible and the real played an important but peculiar role (see e.g. Republic 525d and onwards). In a different philosophical framework it is expressed by Aristotle (see e.g. Metaphysica 1061[a]28-1061[b]33 and 1076[a]37-1078[a]30. esp. 1077[b] onwards; and De caelo 299[a]15-16). It is explained by Hero, according to whom it is the "Wesen (des Punktes) ... nur dem Gedanken fassbar zu sein" (Definitiones 1), and who points more explicitly to the process of idealization in a discussion of the relation between a road and the line through which it is apprehended (ibid. 2). It is not expressed in the Ancient mathematical works themselves, but there is no reason whatsoever to doubt that the mathematicians knew what was expressed in philosophical commentaries to their works";

furthermore, the introductory passages of Archimedes' letter on the Method shows Archimedes fully aware of (and ready to wrestle with) current philosophical viewpoints (see Ver Eecke 1960:478-480).

106 This is of course most obvious in geometry and less so in arithmetic, where the lack of an adequate algebraic formalism would often enforce reasoning through specific numbers. Lots of instances are found by Diophant (Arithmetic, see Ver Eecke 1926). But even here the problems are formulated first in general terms, and only afterwards is the specific numerical example taken in. Just the opposite is found in Egyptian and Mesopotamian mathematics; in both areas general formulations are extremely rare (one dubious instance is the Old Babylonian problem AO 6770 no. 1, cf. Neugebauer 1935a:37-40, Neugebauer 1937:62-63 and van der Waerden 1975:73f; another is Rhind Mathematical Papyrus no. 61b, see Chace 1929).

107 Cf. note 62. It is characteristic that although the three "classical problems" of Greek mathematics (the squaring of the circle, the trisection of the angle, and the doubling of the cube) were solved quite early by means of higher-order curves and so-called "mechanical methods" they continued to occupy the minds because solutions of a different, theoretically more satisfying sort were sought for (see Heath 1921:I,218-270).

Explicit recognition that mathematics was concerned with problems is found by Plato (Republic 530b) and Apollonius (Conica, introductory letter to book IV - Ver Eecke 1963:282, cf. also n. 1). The latter points clearly to two different processes, the construction of the problem being the first, and the discussion of its different solutions the second. Without the word, the thought seems to be present even in Archimedes' introduction to the Method (Ver Eecke 1960:478f).

The primacy of the problem (as compared to the methods used to solve it) is perhaps pointed to by the ethymology of the word and its meaning outside mathematics. It includes "anything put before one", "a barrier", "a task" (Liddell 1975:672).

On "problems" versus "theorems", cf. even Proclus' Commentary 77-78 (Morrow 1970:63-64), and commentary by Zeuthen (1917:37-40). A theorem is an eternal truth; a problem, on the contrary, is a moment of the creative process.

108 The Latin term calculator seems both to have covered the practitioner of calculation and his teacher - for the first meaning, see Kinsey 1979, for the second Bonner 1977:184, and a number of passages in Corpus juris civilis: Digest.XXVII,i,15 §5, Digest.L,xiii,1 §6 (a little dubious) and Codes X,lii,4. Digest XXXVIII,i,7 §5 seems to use the term in the sense of practitioner.

The prestige of the practitioning calculatores was low, viz. (according to the two texts in question) that of slaves and freedmen. Presumably the teacher-calculators fared little better; in any case, the three legal passages precise that such calculatores have no share in the rights of teachers of the liberal arts (cf. below).

Other descendants of the scribal mathematical practitioners were the architects and the tacticians. Both groups were free citizens of considerable prestige; however, for architects we know (Vitruvius, De architectura I,i,3-12) that their mathematical education was just part of a far-ranging liberal education; for tacticians the same will probably hold true (further, their use of mathematics may have been most restricted - a textual passage due to Geminos of Rhodes explaining tacticians use of mathematics may very well speak of armchair tacticians, cf. Aujac 1975:113,163 n.4).

109 The way the Roman Empire organized a system of alimenta (public support for children) exposes its inability to arrange really centralized economic schemes: Payments were not made directly to the cities from the treasury but in an indirect way involving local landowners (see Duncan-Jones 1974:288ff).

110 Metaphysica 981b20.

111 Greek tradition itself certainly ascribed the origin of geometry to Egyptian practitioners, from Herodotus (Histories II, 109) over Strabo (Geography XVII, 3; both passages quoted in Lyons 1926:242) to Proclus (Commentary on the First Book of Euclid's Elements 64-65 - Morrow 1970:52). Aristotle, slightly deviating, ascribes it to the leisure of the Egyptian priestly caste (Metaphysica 981b23). For further related passages, cf. Bretschneider 1870:7ff.

Similarly, Proclus ascribes the origin of arithmetic to the Phoenician traders (Commentary 65 - Morrow 1970:52).

We should, however, be aware that by the time Greek theoretical geometry took its beginning in the sixth century B.C., geometrical practitioners of great skill were at work in Greece itself, as testified e.g. by the immensely precise digging of a tunnel of 1 km on Samos (see Herodotus, Histories III, 60; Sarton 1927:76; and Farrington 1969:43f). Certainly, trade and accounting was also current in Greece in this period.

No doubt Greek practitioners may have learnt a good deal in Egypt and other places (cf. also Hodges 1971:158-160). So, in fact, even when Greek theoretical science speculates about technological practice of a foreign origin we have no way to know whether the theoreticians had got their ideas abroad.

112 Truly, one anecdote ascribes to Thales a piece of directly applied mathematics, viz. the determination of the distance of a ship from shore (see Greek Mathematical Works I, 166f). But as pointed out by van der Waerden (1975:89), the theorems ascribed to Thales form part of a structure of mathematical rationality and proof. So, instead of seeing the supposed

Thales as the inventor or importer of applied mathematics we should probably think of the person (or tradition) in question as someone asking rational questions to the methods of practitioners - in close parallel to what was done to the natural philosophy also ascribed to Thales and the other Milesian philosophers.

113 See Kahn (1974:176) and Knorr (1975:135-137).

114 Cf. Taisbak 1976, and Tannery 1930, 34f.

115 *Laws* VII, 819b-c. The passage in question includes explicitly the partition of a number in aliquot parts in the various possible ways.

116 This connection was more or less explicitly proposed by Vogel (1936:376). Another explanation was proposed by Woepcke (1863:266-273): That the traditional Indian play with huge numbers had been known through Alexander's military conquests. Woepcke's arguments seem rather convincing, but his hypothesis would not exclude that of Vogel.

117 See Ver Eecke 1960:127-134.

118 See e.g. Gandz 1937:416f.

119 *Heronis Alexandrini opera* III, 2ff.

120 *Heronis Alexandrini opera* III, 188ff.

121 This is not the place to investigate the connections between this ideal of education and the two essential sides of the Ancient Greek society: A system of city-states, either governed democratically, by the entire body of free citizens, or aristocratically, by those who according to the meaning of that word had to be *the best*; and based, if not entirely in its social reality then at least in ideals shared by every Greek writer of any importance, on the work of slaves and other non-citizens (cf. Finley 1959, and Greek literature in general). But clearly there were connections.

122 The list of supporting quotations could be long. I shall only refer to Plato, *Republic* 525d, and Aristotle, *Metaphysica* 981^{b} 18-24.

123 It is always difficult to evaluate the influence of ancient authors. Anyhow, there seems to be little doubt that Plato was more read than any other philosopher in Antiquity. Should one doubt the evidence offered by transmitted literature, the papyrus fragments found in Egypt containing Greek literary texts amount to sort of sample inquiry. Plato comes in with 36 fragments, after Homer (555), Demosthenes (74), Euripides (54) and Hesiod (40). Aristotle, on the contrary, is down at 6 fragments (status of 1949, see Finley 1969:17f).

This predominance of Platonic philosophy gives all the more weight to Plato's points of view as expressions of the general philosophic attitude.

124 So, we should remember that the rationalizing spirit in Greek thinking can be traced at least back to Hesiod in the 7th century B.C., well before Thales the supposed father of natural philosophy and mathematics, and *a fortiori* well before the beginnings of the institutionalization of *paideia*. Not only does his *Theogony* bear witness of a desire of systematization (cf. Lloyd 1979:10), a trait after all not very different from what is found with the Babylonians. More decisive, the very introduction to the *Works and Days* (verse 11-51) contains an analysis of the concept "struggle" (êris) by dichotomy - a method typical of Plato and discussed by Aristotle in *Analytica priora* ($46^{a}30$-$46^{b}39$). This early instance of analytical thought should not be dismissed as not concerning mathematics, since only the early fourth century seems to have brought about a relatively clear distinction between "dialectical" and "demonstrative" reasoning, the latter including mathematics, (cf. Lloyd 1979:115ff; Aristotle's critique of certain sophist mathematicians in *Analytica posteriora* $75^{b}40$-$76^{a}3$ and *De sophisticis elenchis* $171^{b}16$-22, $172^{a}3$-7 may be interpreted to hint at the same problem).

125 See Kahn 1974:167f.

126 See on one hand Freeman 1966-75f, on the other Waerden 1979:62f.

127 Plato, *Republic* 600b.

128 Cf. Kahn's discussion (1974).

129 Archytas, Fragment B.1 (Diels 1972:I, 431-435; the relevant passage is 431^{36}-432^{8}.

130 Here I make a maybe somewhat artificial distinction between "philosophical" and "religious" doctrine. To the latter I count the doctrine of the soul, including the transmigration of souls, although of course this touches on philosophical and ontological questions. C.F. Kahn (1974:164ff, 174ff) and Freeman (1966:78).

131 Cf. Freeman (1966:75) and Waerden (1979:66ff).

132 Cf. Archytas, Fragment B.1, "diese Wissenschaften (Geometrie, Arithmetik, Sphärik, Musik) ... beschäftigen sich mit den beiden verschwisterten Urgestalten des Seienden (i.e. Zahl und Grösse)". Diels (1972:I,432^{7-9}).

133 It is difficult to assess the exact contributions of Pythagorean mathematicians, and there may be a tendency to ascribe an early origin to what is in fact the product of much later neo-Pytha-

gorean circles - cf. Diels 1972:I,447 n.3. But it would amount to fanatical criticism of the sources to doubt their far-ranging work in the theory of numbers (the Greek concept of arithmetic - cf. Heath 1921:65-117), connected to their belief in the close connection between numerical relationships and the nature of things (for references, cf. Freeman 1966:246ff). The bulk of definitions (and thereby a large part of the theoretical interests expressed even in the theorems) in Euclid's Elements, books VII-IX, seem to reflect Pythagorean interests. Similarly, the role of the Pythagoreans in the development and probably the creation of mathematics harmonics seems sure (cf. ibid:250f); they seem to have had a sincere interest in astronomy, both observational and numerological (but seemingly not the sort of mathematical astronomy developed by Eudoxus and later mathematicians) (cf. ibid:252f; Heath 1921:162-165). Finally, a number of important geometrical discoveries and research lines are ascribed to them: the sum of the angles of the triangle, the "theorem of Pythagoras", the application of areas, which is the cornerstone of the so-called geometrical algebra; the theory of irrationals as exposed in mature form in the Elements, book X; and the investigation of the stellar pentagon and the five regular solids (cf. Heath 1921:141-162; unpublished work by M. Taisbak points to connections between the pentagram and rather advanced parts of the theory of irrationals, indicating that the Pythagorean circle was important not only for the foundation but even for the high-level developments of that theory).

134 Cf. the phrasing of Archytas, fragment B.1 (Diels 1972:431-435, esp. 431^{36}-432^{8}.

135 Of course, advanced matters like the mature theory of irrationals, if it was really achieved by Pythagoreans, must presumably have exceeded the understanding of the common Pythagorean, and they can hardly have been taught inside a common educational framework. The sources tell us nothing sure about the internal organization of the teaching system of the Pythagoreans, and especially nothing permitting us to assess the degree of personalization of the Pythagorean education (Plato, Phaido 61c, looks slightly like evidence for a rather personal teacher-student relationship, but in relation with Pythagorean ethical attitudes and not with mathematical teachings). So, we cannot know how sharply one should distinguish between one level of institutional teaching aiming at the bulk of mathēmatikoi, providing a fund of basic mathematical knowledge, and another level of individual study where higher levels of the mathematical tradition were transmitted.

136 If this is true, it makes eminently good sense of a passage in Proclus' Commentary 65,
mathematics a "free education"("Paideia eleuthera", as far as I can reconstruct the expression from Ver Eecke's rendering of its single constituents - 1948:57), in a context dealing with

the development of mathematics as a science, not with mathematical education.

137 In fact, the mathēmatikoi were known to be reasoning people, caring for "die Genauigkeit der Argumentation in den mathematischen Wissenschaften ..., weil diese allein Beweise besitzen" (Iamblichus, De communi mathematica scientia 25, a passage probably taken over from Aristotle; quoted from Waerden 1979:332). What distinguishes the mathēmatikoi from what is perhaps another branch of the movement, perhaps a derivation from an outer circle of initiates, the akousmatikoi, is precisely their reasoning and further elaboration of the doctrine, in contrast to the strictly literalist and tradition-bound creed of the latter (cf. Waerden 1979:66-73 and following).

138 An illustrating contrast to the cumulative development of Pythagorean mathematics up to Plato's time is the long survival of banausic numerological semi-mysticism in circles inspired by Pythagoreanism (but hardly descending directly from the original community, cf. Clarke 1971:57, 83) - as found e.g. in Nicomachus' neo-Pythagorean Introduction to Arithmetic. This seems to be the outcome of a sort of mathematical activity which was more religious and literalist than it was reasoning.

It seems inherently plausible that Nicomachean arithmetic is more representative of what could be presented to the ordinary Pythagorean than the theory of prime numbers as found in the Elements, book VII, or the theory of classes of irrational magnitudes (Elements, book X). So, even though the development of Pythagorean theoretical mathematics can be assumed to depend on institutionalized mathematics teaching inside the order we should not believe that its highest level was shared by all mathēmatikoi.

139 In fact, the sophists were probably the creators of an educational system inside the philosophical tradition (cf. Marrou 1956:47). Furthermore, they must be regarded as the creators of the theory of education (cf. Jaeger 1947:I, 298-321).

140 Protagor programme as formulated by Plato (Protagoras 318a, 319a); cf. also the remark on Protagoras et al. in Republic 600c.

141 See Freeman's discussion of the older sophists (1966; esp. pp.343-404, passim). See also the portrait of Socrates in Aristophanes' The Clouds, where he is depicted as one of the sophists.

142 Three sophists are supposed to have contributed to the development of mathematics: Hippias (cf. Freeman 1966: 385-389; and biography in DScB), Antiphon (cf. Freeman

1966:395-397; biography in DScB) and Bryson (cf. biography in DScB). In Plato's Protagoras, Hippias is said (318e, cf. Jaeger 1947:300, 477 n.42) to teach the arts of logistics, astronomy, geometry, and music. Protagoras himself seems at least to have concerned himself with the thought of geometers in so far as to refute them (Aristotle, Metaphysica 998a3-4).

That the probably eminent mathematician Theodorus is presented by Plato as a friend of Protagoras (Theaetetus 161b, 162a) does not count, since Theodorus is made to oppose his former interest in Protagoras' thought with his later interest in mathematics (165a). By the late neo-Platonist Iamblichus he is even identified as a Pythagorean (cf. Diels 1972:I,397[16-17]), an identification, it is true, which may tell more about Iamblichus than about Theodorus (cf. Knorr 1975:5).

143 "Practical" both in the classical sense of "aiming at the good"; and in the modern sense tainted by technical rationality.

144 By the term "discursive" I hint at the history of mathematics perhaps written by Hippias (cf. Bulmer-Thomas in DScB VI, 408). If Hippias wrote such a work (or as also possible a work on the history of philosophy in general - cf. DScB VI, 406) he clearly inaugurated or followed a practice current in ancient philosophy from Aristotle onwards. But nothing similar has been handed down from the hand of any Ancient mathematician.

The term "phenomenalist" alludes to Antiphon's and Bryson's attempts to square the circle by methods which (as far as they can be guessed from the unclear reports given in the sources, cf. Greek Mathematical Works I, 310-317, and biographies in DScB) are built on principles related to those of natural philosophy and common observation of physical circles and tangents, and which are criticized by Aristotle as not in agreement with the specific prinziples of geometry (Analytica priora 75b40-76a3, De sophisticis elenchis 171b16-22, 172a3-7); further I refer to Protagoras' argument against the geometricians that "a hoop touches, a straight edge not at a point (but along a stretch of definite length)" (Aristotle, Metaphysica 998a1-4).

The only piece of sophist mathematics clearly in harmony if not with the main longterm trend in Greek mathematics, then at least with a minor current is Hippias' quadratrix, a transcendental curve used probably by Hippias to trisect the angle (a rather simple matter) and by later geometricians also to square the circle (far less simple) (cf. biography in DScB and Heath 1921:225-230).

145 Source selections in Greek Mathematical Works I, 228-231. Cf. Heath 1921:176-181.

146 So, Democritus is said to have taught Hippocrates of Chios, Protagoras and several others (Freeman 1966:292). Even if the details are unreliable the tale as a whole points to philosophical teaching, as do the similar traditions concerning other philosophers.

147 Aristotle, Metaphysica 998a25-27, cf. 1014a31 - 1014b1. Presumably the degree of axiomatization implied by this definition exceeds that of pre-Platonic Elements. But the connotation of fundamentals and principles from which other entities are composed is probably a true reflection even of late fifth-century Elements. Cf. also Zeuthen (1917:27-29).

148 Proclus, Commentary 66 (Morrow 1970:54). According to Knorr (1975:7), Hippocrates' Elements may have consisted of "an organization of theorems covering Books I and III, Book VI based on a naive proportion concept, and XII, 2 (the measurement of the circle) based on an intuitive limit concept, together with applications not included by Euclid" (all references are to the organization of the Euclidean Elements).

149 Cf. DScB VI, 410.

150 Hippocrates of Chios may have belonged to the supposed "school" of Oenopides of Chios - cf. Bulmer-Thomas in DScB VI, 410. Oenopides, on his part, may be an important figure in the development of Greek mathematics, as being probably the first to restrict the set of geometrical tools to ruler and compass (excluding the gnomon or set square), cf. Zeuthen 1917:64-66, and being engaged in the methodology of mathematics (cf. Bulmer-Thomas in DScB X, 179).

151 That the mainstream of late 5th century mathematics was already both abstract and built on rational argumentation (and thus different from the more phenomenalist geometry of Antiphon and Bryson, cf. n.33.5) is clear from the first piece of Greek geometry handed down in rather undistorted form: Hippocrates' quadrature of lunes (see Greek Mathematical Works I, 234-253, and discussion in Heath 1921:I, 183-200). It also appears from the sorts of problems investigated by contemporary mathematicians (doubling of the cube transferred into the finding of two intermediate proportionals; horn-angles; irrational magnitudes; etc.).

152 In principle, the assertion that axiomatic mathematics did not exist by the mid-fifth century B.C. is a reconstruction, built mainly on the silence of sources. But the form of the argumentation in Hippocrates' treatment of his lunes can at the least be said to support the reconstruction.

153 E.g. Analytica posteriora 76a31-77a4.

154 Greek terms, non-technical translation and standard technical translation of basic categories used in the Euclidean Elements (see Greek Mathematical Works I, 436-445).

155 See Guêraud (1938 - a primary school teacher's manual); Bonner (1977:180-183); and Marrou (1956:157f, 271).

156 From puberty to the age of 21 - cf. Clarke 1971:2, and Aristotle, Politica 1336^{b}37-39.

157 A really fine popularization of astronomy is Geminus' Introduction to the Phaenomena (ed. Aujac 1975; cf. discussion in Heath 1921:II, 232-234).

158 Two instances of such mathematical banality are Nicomachus' Introduction to Arithmetic (ed. d'Ooge 1926), which does not contain a single proof, and Theon of Smyrna's Exposition of the Mathematical Subjects Which Will Be Useful for Reading Plato (ed. Dupuis 1892). Cf. discussions in Heath 1921:I, 97-112; II, 238-244.

159 Ver Eecke 1960:377, 477f. Cf. even pp.3, 137 and especially 239 (Treatise on Spirals).

160 Ver Eecke 1963:1f , 117, 281f (Ver Eecke's identification on p.1 of Apollonius' correspondent Eudemus of Pergamon with Eudemus of Rhodes known for his history of mathematics is impossible for chronological reasons).

161 Arithmetic, book I - Ver Eecke 1926:1.

162 Mathematical Collection, books 3, 5, 7 and 8 - Ver Eecke 1933:21f, 237, 477 and 809. The reserve concerns book 7 and 8 written to "mon fils Hermodore".

163 The introduction to the commentary on Archimedes' Treatise on the Equilibrium of Planes - Ver Eecke 1960:721. The introduction to his commentary on Archimdes' Treatise on the Sphere and Cylinder (Ver Eecke 1960:555f), on the other hand, is very personally directed to his teacher Ammonius son of Hermias.

164 Archimedes to Gelon, in the Sandreckoner (Ver Eecke 1960:353). Eratosthenes to king Ptolemy, quoted by Eutocius (Ver Eecke 1960:609).

165 According to Plato (Laws 820b) one ought to blush from shame if erring in the newly developing theory of incommensurability. Plato's contemporary Isocrates recommends for the adolescent a suitable amount of mathematical training, as both manly and fit to train the mind (see Heath 1921:I,21). At least the first term suggests "vigour" and "success" as connotations linked in the mind of Isocrates to the conception of mathematical activity.

166 Clarke 1971:59.

167 Cf. a friendly satire from a comedy, retold by Clarke (1971:66).

168 819b-c.

169 Proclus, Commentary 67 - Morrow 1970:56.

170 Commentary 67f - Morrow 1970:55f.

171 Proclus' main source is a history of mathematics written by Eudemus of Rhodes, who was a younger contemporary of Aristotle and so of the group of mathematicians in question; his account must thus be considered to be reliable. Concerning Eudoxus, furthermore, Proclus' somewhat unclear statements are illuminated and strengthened by independent evidence, e.g. on his role as the creator of the general theory of proportions (whether in its Euclidean form, cf. Heath 1921:I,325, or in a precursor version, cf. Knorr 1978) and the method of exhaustion (cf. Heath 1921:I, 327-329).

172 Such "schools" are attributed to Menaechmus, Speusippus (Plato's successor at the head of the academy) and Amphinomus (Proclus, Commentary 77-78, 254 - Morrow 1970:63f, 197). The Greek expressions (oî peri Mênaichmon mathēmatikoî, cf. Zeuthen 1917:38, and corresponding formulations) are equivalent to "the mathematicians around Menaechmus", and so point to an informal circle, not necessarily to any kind of institutionalization.

173 E.g. Zeuthen (1917); Solmsen (1931); and Dehn (1936).

174 Cf. Clarke (1971:66).

175 If so, Plato's philosophical requirements to mathematics would be an intermediate factor, formulating explicitly (and thereby perhaps strengthening) the pre-existent principles directing the development of mathematics (cf. also Zeuthen 1917:34).

176 Proclus, whose Commentary on the First Book of Euclid's Elements I have quoted many times, was the head of the Academy until his death in 485 (Sarton 1927:402).

177 See Clarke 1971:61-69, passim.

178 Ibid:69. Cf. also Lloyd 1973:8-20.

179 On all these institutions, see ibid:55-108, "Philosophical teaching".

180 Wussing 1965:124.

181 Cf. biography in DScB X, 293ff.

182 Cf. biographies in DScB XIII, 321ff (Theon) and DScB VI, 615f (Hypatia).

183 Cf. DScB IV, 414. A recent proposal that Euclid may have been contemporary with Archimedes (Schneider 1979:61f n.82) does not concern us here, as it changes nothing in subsequent arguments.

184 So, by Knorr (1978) and Schneider (1979:61f n.82).

185 "... books that all the practitioners of a given field knew intimately and admired, achievements upon which they modelled their own research and which provided them with a measure of their own accomplishment" (Kuhn 1961:352).

186 For the mainly custodian character of Syriac learning, cf. what Sarton (1927:611 and passim) has to say on the subject of pre-Islamic Syriac science. As to Pehlevi learning, cf. Pingree 1963:241-246.

187 Far Eastern traditions (China, Korea, Japan) I omit for two reasons. First, I know next to nothing about them. Second, their influence on modern mathematics seems to be very restricted and indirect.

188 This early Indian geometry is known from a number of texts datable to the mid-first millenium B.C., dealing with the sacred rules for constructing altars (see Sen 1971:139f, 143f, 145-156).

189 Early Vedic texts contain numbers up to 10^{12}, interwoven into texts on the values of various sacrifices (see Sen 1971:141). Buddhists and Jainists went even further in this way, not only to still higher powers of 10 (ibid:141) but even into constructions related to Archimedes' Sand-Reckoner and into conceptions of infinities of different orders (cf. Woepcke 1863:255-266; and Sen 1971:159).

190 Sen (1971:157ff; 1971a:59,80f).

191 For Jaina and later mathematics one may also suspect a commercial function or background. The prospering commercial communities from 200 B.C. onwards supported Buddhism and Jainism (Thapar 1966:109). No Jainist mathematical texts from the early centuries have been handed down, but the same combination of arithmetical-algebraic and astronomical interests as found in the Jaina canon occurs in later works still extant: The Aryabhatīya (c. A.D. 500; see Clark 1930; and description in Sen 1971a:93f); the Brāhmasphuta-siddhānta (c. 600; mathematical chapters in Colebrooke 1817; description in Sen 1971a:95f); and, still earlier, the "Bakhshālī manuscript" (c. 300? see Hoernle 1883 and 1888). In all cases, commercial problems are found in the arithmetical introduction to astronomy (in the early Bakhshālī manuscript they dominate completely). The mixture of (sacrally oriented) astronomy and commercial calculation points to a mixed motivation for mathematical activities, and the continuities suggested by the various sources makes it plausible that this merger can be projected backwards in time to the early Jaina centuries.

192 The early Vedic geometrical texts are all the way through formulated directly as practical rules for constructions of the altars required in the older texts on sacrifices to which they were attached (see Seidenberg 1962:506ff, and the texts as published in Bürk 1901 and Thibaut 1874). For the ritual use of astronomy, see Sen (1971:59): Even astronomy concerned sacrifice, as it was used for the fixation of the time when sacrifices should be performed.

On the immense elaboration of Hindu ritual, cf. Staal 1979.

193 Altekar (1959:424). Given the lack of written sources, the term "everybody" as well as other details must be taken with some caution. Cf. also Thapar 1966:42.

194 Enc.Brit. 9:319. Thapar (1966:37-40). Kosambi (1970:209-211).

195 Thapar (1966:44).

196 Thapar (1966:42). Enc. Brit. 9:319. Altekar (1959:425). Kosambi (1970:206).

197 Altekar 1959:429.

198 1971:140.

199 See Bürk 1901:I, 572f; II, 334f.

200 See Pingree (1963:234-239) and Sen (1971a:81ff).

201 This is the case in the Bakhshālī manuscript (Hoernle 1888) and in the 12th century Līlāvatī (in Colebrooke 1817:1-127). In the 7th century Brāhmasphuta-siddhānta they are lacking, but integrated into the text by a later commentator (in Colebrooke 1817:277-378; cf. note on p. 278).

202 Such "proofs" are given in the Bakhshālī manuscript.

203 Retold by Neugebauer (1952:253).

204 Altekar 1959:429.

205 Not least the politico-religious contradictions which in the end materialized in the main tension between shī'a and sunna, but where the subdivisions involving ismā'īlīsm, sūfīsm and

mu'tazilism play a role affecting the teaching and the study of mathematics; cf. surveys by Gardet (1970), Cahen (1968: 208-222), Arberry (1970) and Hvidtfeldt (1962:210-234). Cf. also Nasr (1968:90f), and Heinen (1978).

Also the contradiction between "traditional" or religiously bound learning, and "rational" or "pre-Islamic" (i.e. mainly Greek) learning and philosophy is important; cf. ibn Khaldūn, Muqaddimah VI, 9 (Rosenthal 1958:II, 436ff) and, for modern discussions, Fakhry (1969:91), Nasr (1968:70-74) and Anawati (1970:745f).

Finally the division of the institutions of learning in two more or less separate groups is influential. On one hand there are the mosque-dependent institutions (the normal madrasa's), on the other those not directly submitted to the mosque (lay institutions in the Medieval meaning of that word, where even laity was in the final instance submitted to church and religion): Libraries eventually provided with a staff of scholars; hospitals; and observatories. On the madrasa and its relatives, cf. Makdisi (1961; 1970; 1971; 1971a). On hospitals, Nasr (1968:89) and Anawati (1970:749). On observatories Sayili (1960) and Nasr (1968:80ff). On libraries and libraries cum academy, see (for various aspects and various institutions) Pines (1970:783f), Anawati (1970:748f), Makdisi (1961:7f), Nasr (1968:69f) and Juschkewitsch (1964:184).

The whole development of mathematics and teaching is woven into this network of contradictions and divisions.

206 By the expression "Islamic mathematics" I refer to mathematics belonging to the Islamic world. It will be noticed that quite a few "Islamic mathematicians" were Jews, Christians or Sabians (belonging to what seems to be a syncretistic Greek-Babylonian religion, strongly tainted by neo-Pythagoreanism - cf. Waerden 1979:319f).

207 Full and detailed argumentation for this trichotomy would require much space. It is founded partly on mathematical substance; partly on connections established by common treatment of certain mathematical subjects in fixed book traditions in certain institutional settings; and partly on common application. Some of the arguments will turn up in the following.

Certain subjects fall on the margin of the three categories enumerated here: Magical squares; the "mathematics of optics".

208 The oldest text known (but not necessarily the first of its kind) is that by al-Uqlīdisī (translation Saidan 1978). Strictly speaking, it is perhaps more of an immensely extended algorism than a real "Rechenbuch". If we disregard it, the On what scribes, officials, and others need of the science arithmetic by Abū'l-Wafa' from c. 970 will be the oldest still existing specimen (cf. Saidan 1974 and Juschkewitsch in DScB I, 39). The latest specimen which I have looked at was written by al-Qalasādī in the mid-fifteenth century, i.e. in the very end of the Middle Ages (translation Woepcke 1859).

209 See Cohen (1970, esp. pp.35ff). Cohen's material is most convincing from the ninth century onwards (the period, incidentally, where Islamic mathematics rose). That, however, is due to the poverty of earlier material; so, the intimate connection between religious learning and commerce and crafts may very well be of older date.

210 Cf. review article by Hoffmann (1974).

211 The finger-reckoning was called "arithmetic of the Byzantines and the Arabs" (cf. Saidan 1974:367), and was certainly known by Greco-Roman practitioners (cf. note 86). The Pythagorean definition of multiplication (Nicomachus, Introduction I, xviii - d'Ooge 1926:214; Euclid, Elements VII, def. 5 - Heath 1956:II, 277) is used in Abū'l-Wafā's On what scribes ... which is based exclusively on finger- and traditional verbal numeration (Saidan 1974:370). Greek neo-Pythagorean arithmetic had also knowledge of and interest in the summation of series (cf. e.g. Nikomachus, Introduction II, vii-II, xiv - d'Ooge 1926:241-252; similar passages in Theon of Smyrna's Exposition). Even late Babylonian mathematics was interested in the summation of series (an interest which may have been taken over from contact with the Greeks - cf. however alternative explanations in note 85).

212 On interest in magical squares in Islam, see Ahrens (1916), Bergsträsser (1923), Hermelink (1958; 1959) and Sarton (1927:661, 753; 1931:187, 596, 600, 624, 1000). Stapleton's paper (1953) is better left out as speculative.

It has been maintained that magical squares are of neo-Pythagorean origin, and especially it has been said that Theon of Smyrna's Exposition II, xliv (Dupuis 1892:167f) suggests aquaintance with magical squares (e.g. Sarton 1927:272). I would rather say that the passage in question proves almost beyond doubt that Theon did not know that concept - he is so close that he would certainly have mentioned the thing had he known it. So, of neo-Pythagorean in origin, magical squares were invented in later Antiquity or the early Middle Ages.

213 See Cohen (1970:41).

214 If we disregard books written directly as propaganda for the Hindu methods (the short treatise by al-Khwārizmī which may have been the first - translations Vogel 1963 and Boncompagni 1857; al-Uqlīdisī's book - translation Saidan 1978; and Kushyār ibn Labbān's compendium for astronomers - translation Levey 1965), then the first arithmetic books written for the use of practitioners are based on the Arabic verbal number system and on finger-reckoning (so, that by Abū'l-Wafā' - summary Saidan 1974; and that by al-Karajī - surveys by Saidan 1978:19f,

and Benedict 1914:7f). Later works (so that of al-Qalasādī - translation Woepcke 1859) have changed completely to Hindu numbers, even if they are clearly aimed at least indirectly towards a practitioners' public, as demonstrated e.g. by their ample treatment of the traditional system of Arabic fractions still used by non-scientific practitioners.

Cf. also Juschkewitsch in DScB I, 40 (biography of Abū'l-Wafā'), and Anbouba in DScB XII, 94 (biography of al-Samaw'al).

215 On the Arabic fractions, see Saidan (1974:368f), samt de i note 214 omtalte biografier af Abū'l-Wafā' og al-Samaw'al. In contrast to what happened to numerals, the general fractions never expelled the traditional ones completely.

216 Cf. biographies of al-Bannā, al-Qalasādī and al-Umawī in DScB (I, 437; XI, 229; and XIII, 539).

217 The close dependence of Leonardo's Liber abaci on the Islamic Rechenbücher hardly needs to be argued. It is obvious to the first glance at the text, and it is commonly recognized. The intimate connection between Jordanus' algebra (De numeris datis) and Islamic algebra is less recognized, but can be proven from Jordanus choice of numerical examples identical with those of the Islamic texts.

218 Anbouba (1978:74f).

219 The summation $1^2 + 2^2 + 3^2 + \ldots + 10^2$ already mentioned in the late Babylonian text AO 6484 (cf. note 85) is found by al-Hassār (see Suter 1901:33), solved by the same "formula" out of several possible.

220 See Nasr (1968:80 n.5).

221 Maybe by Theon of Alexandria, cf. DScB IV, 430.

222 Cf. Steinschneider (1865), from where the list is taken (pp. 462f and 467).

223 See Steinschneider 1865:459. A real fixation of the curriculum only took place in the mid-thirteenth century, when Nasir al-Dīn al-Tusī's revisions and commentaries were accepted to such a degree that it amounts to a real institutionalization. Cf. also DScB XIII, 508 (Nasr, biography of Nasir al-Dīn).

224 That the main part of this corpus did in fact function as an autonomous mathematics course appears also from al-Khayyāmī's list of works which he presupposes as basic knowledge by the readers of his Algebra (translation Kasir 1931): Euclid's Elements and Data, Apollonius' Conica I-II and (not mentioned explicitly on p.48 but clearly presupposed in the following text) the established algebraic tradition.

225 So, both ibn al-Haytham and al-Khayyāmī wrote on the solution of foundational "difficulties" in Euclid's Elements (see DScB VI, 207 nr. III,39 and 208 nr. Add. 1 for ibn al-Haytham, and Amir-Môez for a translation of al-Khayyāmī's work; the latter refers back to ibn al-Haytham's work on p.277).

226 So the phrasing of al-Khayyāmī's introduction (Amir-Móez 1959:276f).

227 Already in 1159, this connection between astronomy and Islamic pure mathematics (to be more explicit, pure geometry) was suggested by a Western Latin observer: John of Salisbury, in Metalogicon IV, 6. Discussing the use of Aristotle's Analytica posteriora, he writes, that "At present demonstration is employed by practically no one except mathematicians, and even among the latter has come to be almost exclusively reserved to geometricians. The study of geometry is, however, not well known among us, although this science is perhaps in greater use in the region of Iberia and the confines of (the Ancient Roman province) Africa. For the peoples of Iberia and Africa employ geometry more than do any others; they use it as a tool in astronomy. The like is true of the Egyptians, as well as some of the peoples of Arabia" (translation McGarry 1971:212).

228 The annotation in the following chapter will be reduced to the barest minimum - mostly identification of sources. Underpinning of generalizations will be omitted for a very trivial reason: At the time when the editor of the series had to ask me for a binding estimate of the extension of my essay I was very far from having finished it. My estimate proved wrong. So, this chapter will have to be read either on faith or, better, as a sketch combining established beliefs (which I share) with facts, theories and assumptions taken over from other authors and with hypotheses and generalizations of my own: A sketch for further research.

229 Boethius, De musica III, xi (PL 63, Col. 1236f).

230 Isidor of Sevilla, Etymologiae III, iv (PL 82, col. 156; translation JH). Isidor was one of the most widely read and popular authorities throughout the Middle Ages.

231 Al-Uqlīdisī (Saidan 1978:313f) describes a device closely related to the very peculiar "Gerbert-abacus". Together with internal evidence pointing to inspiration from the Islamic West this suggests import and not rediscovery.

232 For discussion of this point, see Ullman 1964.

233 Bubnov (1899:43-45) and Tannery (1922) give some texts.

234 Quotation in Grabman (1941:61).

235 Murdoch in DScB IV, 444. Description of all 12th century versions pp. 444-447, and more fully in Murdoch (1968).

236 E.g. in the beginning of book V (Euclidis Megarensis ... 103-106).

237 Petrus Philomenus de Dacia, in Curtze (1897:20).

238 Opus Tertium, cap. vi; quoted in Smith 1914:163. It should be noted that Bacon's judgment of the mathematical abilities of his contemporaries is in general as unreliable as his own understanding of the best of contemporary mathematics is poor.

239 Grabman (1934:218).

240 E.g. Denifle (1889:277-279, 78) and S. Gibson (1931:33).

241 Haskins (1929:47).

242 Algorismus demonstratus, published by Eneström (1912; 1913).

243 1889:17.

244 Cf. Murdoch (1969).

245 As already noticed by Pomponazzi in 1514 (quotation in Wilson 1953:360).

246 The existence of a separate tradition was suggested by Cantor (1900:166) and denied with twisted readings and usual acrimony by Eneström (1906). Cantor's main material was a couple of 14th and 15th manuscripts, selections of which were published by Libri (1838:III, 302-356). Recent publications led support to Cantor's idea (Jayawardene 1976; Davis 1977; a number of relevant publications mentioned by Jayawardene I have not seen). Important is also Karpinski (1910).

247 See Jayawardene (1976:233-235).

248 Summa de arithmetica, quoted by Marre (1880:568).

249 For comment and printed edition, see Marre (1880; 1880a). On Italian influence, see Marre (1880:566).

250 Vogel (1964: 1954).

251 So, the divergent search for and the resulting improvements in mathematical notation, expressed not least in books descending from the abacus-school tradition, I would rather ascribe to the organization of mathematical communication

around such printed books than to any sort of didactical organization - even though, I must confess, this is only a guess derived from impressions.

252 Paraphrasing Jahnke's title (1978).

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